Brief Notes #6

Second-Moment Characterization of Random Variables and Vectors. Second-Moment (SM) and First-Order Second-Moment (FOSM) Propagation of Uncertainty

(a) Random Variables

- **Second-Moment Characterization**

  - *Mean (expected value) of a random variable*

$$E[X] = m_X = \sum_{\text{all } x_i} x_i P_X(x_i) \qquad \text{(discrete case)}$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)dx \qquad \text{(continuous case)}$$

  - *Variance (second central moment) of a random variable*

$$\sigma_X^2 = \text{Var}[X] = E[(X - m_X)^2] = \sum_{\text{all } x_i} (x_i - m_X)^2 P_X(x_i) \qquad \text{(discrete case)}$$

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x - m_X)^2\, f_X(x)dx \qquad \text{(continuous case)}$$

  - *Examples*

    – Poisson distribution

$$P_Y(y) = \frac{(\lambda t)^y e^{-\lambda t}}{y!} \qquad y = 0, 1, 2, \ldots$$

$$m_Y = \lambda t$$

$$\sigma_Y^2 = \sum_{y=0}^{\infty} (y - \lambda t)^2 P_Y(y) = \lambda t = m_Y$$

- Exponential distribution

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0$$

$$m_X = \frac{1}{\lambda}$$

$$\sigma_X^2 = \int_0^\infty (x - \frac{1}{\lambda})^2 f_X(x)\, dx = \left(\frac{1}{\lambda}\right)^2 = m_X^2$$

- *Notation*

  $X \sim (m, \sigma^2)$ indicates that X is a random variable with mean value m and variance σ^2.

- *Other measures of location*

  - Mode $\tilde{x}$ = value that maximizes P_X or f_X

  - Median x_{50} = value such that $F_X(x_{50}) = 0.5$

- *Other measures of dispersion*

  - Standard deviation

    $$\sigma_X = \sqrt{\sigma_X^2} \qquad \text{(same dimension as X)}$$

  - Coefficient of variation

    $$V_X = \frac{\sigma_X}{m_X} \qquad \text{(dimensionless quantity)}$$

- **Expectation of a Function of a Random Variable. Initial and Central Moments**

  - *Expected value of a function of a random variable*

    Let Y = g(X) be a function of a random variable X. Then the mean value of Y is:

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} y\, f_Y(y) dy$$

Importantly, it can be shown that E[Y] can also be found directly from f_X, as:

$$E[Y] = \int_{-\infty}^{\infty} g(x)\, f_X(x) dx$$

- *Linearity of expectation*

  It follows directly from the above and from linearity of integration that, for any constants a_1 and a_2 and any functions $g_1(X)$ and $g_2(X)$:

  $$E[a_1 g_1(X) + a_2 g_2(X)] = a_1 E[g_1(X)] + a_2 E[g_2(X)]$$

- *Expectation of some important functions*

  1. $E[X^n] = \int_{-\infty}^{\infty} x^n\, f_X(x) dx$

     (called *initial moments*; the mean m_X is also the ***first initial moment***)

  2. $E[(X - m_X)^n] = \int_{-\infty}^{\infty} (x - m_X)^n\, f_X(x) dx$

     (called *central moments*; the variance σ_X^2 is also the *second central moment*)

- **Consequences of Linearity of Expectation. Second-Moment (SM) Propagation of Uncertainty for Linear Functions**

  1. $\sigma_X^2 = Var[X] = E[(X\text{-}m_X)^2] = E[X^2] - 2m_X E[X] + m_X^2 = E[X^2] - m_X^2$

     $\Rightarrow E[X^2] = \sigma_X^2 + m_X^2$

  2. Let Y = a + bX, where a and b are constants. Using linearity of expectation, one obtains the following expressions for the mean value and variance of Y:

     $$m_Y = a + bE[X] = a + bm_X$$

     $$\sigma_Y^2 = E[(Y - m_Y)^2] = b^2 \sigma_X^2$$

- **First-Order Second-Moment (FOSM) Propagation of Uncertainty for Nonlinear Functions**

  Usually, with knowledge of only the mean value and variance of X, it is impossible to calculate m_Y and σ_Y^2. However, a so-called first-order second-moment (FOSM) approximation can be obtained as follows.

  Given $X \sim (m_X, \sigma_X^2)$ and $Y = g(X)$, a generic nonlinear function of X, find the mean value and variance of Y.

  → Replace g(X) by a linear function of X, usually by linear Taylor expansion around m_X. This gives the following approximation to g(X):

  $$Y = g(X) \approx g(m_X) + \left.\frac{dg(X)}{dX}\right|_{m_X} (X - m_X)$$

  Then approximate values for m_Y and σ_Y^2 are:

  $$m_Y = g(m_X), \qquad \sigma_Y^2 = \left(\left.\frac{dg(X)}{dX}\right|_{m_X} \right)^2 \sigma_X^2$$

(b) <u>**Random Vectors**</u>

- **Second-Moment Characterization. Initial and Central Moments**

  Consider a random vector $\underline{X}$ with components $X_1, X_2, \ldots, X_n$.

  - *Expected value*

    $$E[\underline{X}] = E\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_n] \end{bmatrix} = \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = \underline{m} \qquad \text{(mean value vector)}$$

  - *Expected value of a scalar function of* $\underline{X}$

    Let $Y = g(\underline{X})$ be a function of $\underline{X}$. Then, extending a result given previously for function of single variables, one finds that E[Y] may be calculated as:

$$E[Y] = \int_{R^n} g(\underline{x})\, f_{\underline{X}}(\underline{x}) d\underline{x}$$

Again, it is clear that linearity applies, in the sense that, for any given constants a_1 and a_2 and any given functions $g_1(\underline{X})$ and $g_2(\underline{X})$:

$$E[a_1 g_1(\underline{X}) + a_2 g_2(\underline{X})] = a_1 E[g_1(\underline{X})] + a_2 E[g_2(\underline{X})]$$

- *Expectation of some special functions*

  - Initial moments

    1. Order 1: $E[X_i] = m_i \quad \Leftrightarrow \quad E[\underline{X}] = \underline{m} \qquad i = 1, 2, \ldots, n$

    2. Order 2: $E[X_i X_j] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i x_j\, f_{X_i,X_j}(x_i, x_j) dx_i dx_j \qquad i, j = 1, 2, \ldots, n$

    3. Order 3: $E[X_i X_j X_k] = \ldots \qquad i, j, k = 1, 2, \ldots, n$

  - Central moments

    1. Order 1: $E[X_i - m_i] = 0 \qquad i = 1, 2, \ldots, n$

    2. Order 2 (*covariance* between two variables):

       $Cov[X_i, X_j] = E[(X_i - m_i)(X_j - m_j)] \qquad i, j = 1, 2, \ldots, n$

       $$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_i - m_i)(x_j - m_j)\, f_{X_i,X_j}(x_i, x_j) dx_i dx_j$$

  - Covariance in terms of first and second initial moments

    Using linearity of expectation,

    $$Cov[X_i, X_j] = E[(X_i - m_i)(X_j - m_j)] = E[X_i X_j - X_i m_j - m_i X_j + m_i m_j]$$

    $$= E[X_i X_j] - m_i m_j$$

    $$\Rightarrow E[X_i X_j] = Cov[X_i, X_j] + m_i m_j$$

- **Covariance Matrix and Correlation Coefficients**

  - *Covariance matrix*

$$\underline{\Sigma}_{\underline{X}} = \begin{bmatrix} Cov[X_i, X_j] & & \\ & \ddots & \\ & & (i, j = 1, 2, ..., n) \end{bmatrix}$$

$$= E[[\underline{X} - \underline{m}_{\underline{X}}]\,[\underline{X} - \underline{m}_{\underline{X}}]^T]$$

  - For n = 2:

$$\underline{\Sigma}_{\underline{X}} = \begin{bmatrix} \sigma_1^2 & Cov[X_1, X_2] \\ Cov[X_1, X_2] & \sigma_2^2 \end{bmatrix}$$

X2, X1

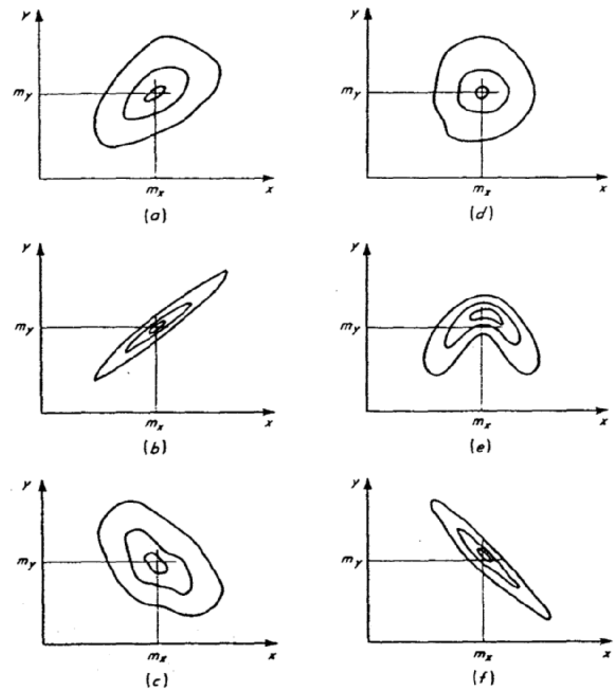
  - $\underline{\Sigma}_{\underline{X}}$ is the matrix equivalent of σ_X^2
  - $\underline{\Sigma}_{\underline{X}}$ is symmetrical: $\underline{\Sigma}_{\underline{X}} = \underline{\Sigma}_{\underline{X}}^T$

  - *Correlation coefficient between two variables*

$$\rho_{ij} = \frac{Cov[X_i, X_j]}{\sigma_i \sigma_j} \qquad i, j = 1, 2, \ldots, n, \qquad -1 \le \rho_{ij} \le 1$$

  - ρ_{ij} is a measure of the linear dependence between two random variables;
  - ρ_{ij} has values between -1 and 1, and is dimensionless.

X1, X2



Joint density-function contours of correlated random variables. (a) Positive correlation $\rho > 0$; (b) high positive correlation $\rho \approx 1$; (c) negative correlation $\rho < 0$; (d) (e) low correlation $\rho \approx 0$; (f) large negative correlation $\rho \approx -1$.

- **SM Propagation of Uncertainty for Linear Functions of Several Variables**

Let $Y = a_0 + \sum_{i=1}^{n} a_i X_i = a_0 + a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$ be a linear function of the vector $\underline{X}$. Using linearity of expectation, one finds the following important results:

$$E[Y] = E\left[a_0 + \sum_{i=1}^{n} a_i X_i\right] = a_0 + \sum_{i=1}^{n} a_i m_i$$

$$Var[Y] = \sum_{i=1}^{n} a_i^2 Var[X_i] + 2\sum_{i=1}^{n} \sum_{j=i+1}^{n} a_i a_j Cov[X_i, X_j]$$

- *For n = 2:*

$$Y = a_0 + a_1 X_1 + a_2 X_2$$

$$E[Y] = a_0 + a_1 E[X_1] + a_2 E[X_2]$$

$$Var[Y] = a_1^2 Var[X_1] + a_2^2 Var[X_2] + 2\, a_1 a_2 Cov[X_1, X_2]$$

- *For uncorrelated random variables:*

$$Var[Y] = \sum_{i=1}^{n} a_i^2 Var[X_i]$$

- *Extension to several linear functions of several variables*

Let $\underline{Y}$ be a vector whose components Y_i are linear functions of a random vector $\underline{X}$. Then, one can write $\underline{Y} = \underline{a} + \underline{B}\ \underline{X}$, where $\underline{a}$ is a given vector and $\underline{B}$ is a given matrix. One can show that:

$$\underline{m_Y} = \underline{a} + \underline{B}\ \underline{m_X}$$

$$\underline{\Sigma_Y} = \underline{B}\ \underline{\Sigma_X}\ \underline{B}^T$$

- **FOSM Propagation of Uncertainty for Nonlinear Functions of Several Variables**

Let $\underline{X} \sim (\underline{m_X}, \underline{\Sigma_X})$ be a random vector with mean value vector $\underline{m_X}$ and covariance matrix $\underline{\Sigma_X}$. Consider a nonlinear function of $\underline{X}$, say $Y = g(\underline{X})$. In general, m_Y and σ_Y^2 depend on the entire joint distribution of the vector $\underline{X}$. However, simple approximations to m_Y and σ_Y^2 are obtained by linearizing $g(\underline{X})$ and then using the exact SM results for linear functions. If linearization is obtained through linear Taylor expansion about $\underline{m_X}$, then the linear function that replaces $g(\underline{X})$ is:

$$g(\underline{X}) \approx g(\underline{m}_X) + \sum_{i=1}^{n} \left. \frac{\partial g(\underline{X})}{\partial X_i} \right|_{\underline{X}=\underline{m}_X} (X_i - m_i)$$

where m_i is the mean value of X_i. The approximate mean and variance of Y are then:

$$m_Y = g(\underline{m}_X),$$

$$\sigma_Y^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} b_i b_j \mathrm{Cov}[X_i, X_j]$$

$$\text{where } b_i = \left.\frac{\partial g(\underline{X})}{\partial X_i}\right|_{\underline{X}=\underline{m}_X}$$

This way of propagating uncertainty is called *FOSM analysis*.