1.061 / 1.61 Transport Processes in the Environment
Fall 2008

Answer 10.1
First, we find the particle settling velocity, w_P. Assuming $Re_P < 1$, then

$$w_P = \frac{gd^2(\rho_P - \rho_F)}{18\mu_F} = \frac{9.8ms^{-2}\left(62 \text{ x } 10^{-6} m\right)^2 (2600 - 1000)kgm^{-3}}{18(1000 \text{ kgm}^{-3})(10^{-6} m^2 s^{-1})} = 0.0033 ms^{-1}.$$

We check our assumption: $Re_P = (62 \text{ x } 10^{-6} m)(0.0033 ms^{-1})/(10^{-6} m^2 s^{-1}) = 0.2 < 1$. OK.

Second, we determine if the system is laminar or turbulent in order to pick an appropriate settling model. The mean velocity is $u = Q/(b\ h) = 0.03 m^3 s^{-1}/(2m \text{ x } 1m) = 0.015\ ms^{-1}$. From this we estimate the friction velocity $u_* \approx 0.1u = 0.0015 ms^{-1}$. Since $u_* < w_P$, settling will more closely follow the laminar or slow-mixing model. Specifically, with weak mixing the concentration in the water column remains constant even as mass is lost to the bed through settling. We may additionally assume that longitudinal mixing is weak, and thus use a plug-flow type model (see Chapter 2). The flow is considered to be a series of thin fluid slabs, each of longitudinal thickness dx. As each slab enters it contains suspended particles at concentration C_O. As the slab moves downstream at speed u, mass is lost from the slab through settling at the rate $\partial M/\partial t = -w_P C_o b dx$. If the slab enters the pond at t = 0 with initial mass $M_o = C_o h b dx$, the mass remaining in the slab for t > 0 is

$M(t) = C_o hbdx - (w_P C_o bdx)t$,

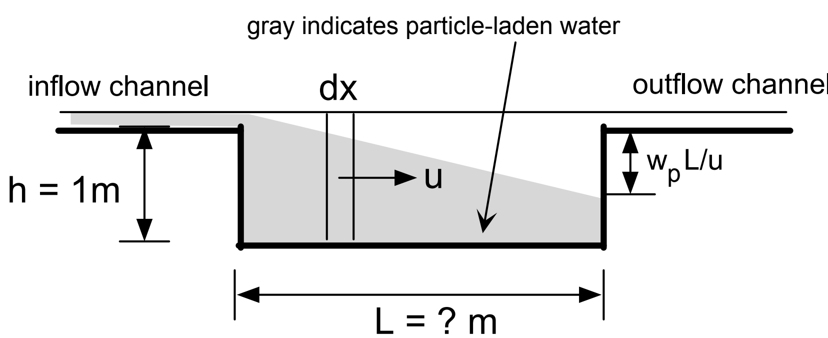
which can also be written,

$(M/M_o) = 1 - (w_P/h)t$.

The slab reaches the outflow at t = L/u, at which time we would like $M/M_O = 0.15$. Using these values we can solve for the required L.

$$\frac{M}{M_o} = 1 - \frac{w_p}{h}\frac{L}{u}$$

which gives $$L = \frac{h\ u\ (1 - M/M_o)}{w_P} = \frac{(1m)\ (0.015 ms^{-1})(1 - 0.15)}{0.0033 ms^{-1}} = 3.9 \text{ m}$$



The above picture shows how the region of particle-laden water evolves across the pond. The picture suggests that if outflow is drawn from the surface, it will be free of particles, *i.e.* the effective removal can be 100%.

Answer 10.2

Start by estimating the settling velocity of the particles assuming creeping flow.

$$w_P = \frac{gd^2(\rho_P - \rho_F)}{18\mu_F} = \frac{9.8\text{ms}^{-2}(10^{-5}\text{m})^2(2000 - 998\text{ kgm}^{-3})}{18(10^{-3}\text{kgm}^{-1}\text{s}^{-1})} = 5.5 \times 10^{-5}\text{ms}^{-1}$$

Check assumption of creeping flow. $Re = \rho_F w_P d/\mu_F = 0.0005 < 1$, so creeping flow assumption is OK. Next, find the time-scale at which the center of mass of the particle cloud will settle.

$T_{settle} = h/w_s = 0.05 \text{ m} / (5.5 \times 10^{-5} \text{ ms}^{-1}) = 910 \text{ s}$.

With this time scale and the mean advection, u = 0.1 cm/s (assumed constant over channel depth), the center of the cloud will settle at a distance $x = uT_{settle} = 91$ cm downstream from the source. The footprint of the particle cloud, *i.e.* its length and width, depends on the size of the cloud as it meets the bed. While in suspension, the particles spread in all directions by isotropic diffusion, $D = 10^{-13}\text{ m}^2\text{s}^{-1}$. Neglecting the boundary for a moment, at $t = T_{settle}$ the particle cloud will be spherical with diameter $\approx 4\sigma = 4\sqrt{(2 \times 10^{-13}\text{m}^2\text{s}^{-1} \times 910 \text{ s})} = 5.4 \times 10^{-5}$ m. Because the cloud is distributed vertically over a distance $\approx 4\sigma$, individual particles may settle slightly before or after the mean settling time given above. The range of settling times is $\Delta T = 4\sigma/w_s = 1$ s. With advection u = 0.1cm/s, the range of settling times will spread the particle patch longitudinally over a distance, $\Delta T\, u = 0.1$ cm. In addition, the cloud is already distributed over a longitudinal distance 4σ as it settles. But, the longitudinal spread due to diffusion while in suspension, $4\sigma = 0.005$ cm, is small compared to the longitudinal spread accomplished by the differential settling times (0.1 cm), so we ignore the former. Thus we expect the footprint of the particle cloud to be 0.1 cm long, 0.005 cm wide, and centered 91 cm downstream of the release. Given that the longitudinal distance traveled (91 cm) is much greater than the distribution caused by diffusion (0.1 cm), to first order the particles' fate is determined solely by the settling velocity and current speed, i.e. by advection. We could have predicted this apriori by considering the Peclet number based on the particle settling speed. $Pe = w_P h/D = 3\times10^7 >> 1$, indicating that advective transport dominates in this case.

Answer 10.3

Start by estimating the fall velocity of the seeds assuming creeping flow

$$w_P = \frac{gd^2(\rho_P - \rho_F)}{18\mu_F} = \frac{9.8\text{ms}^{-2}(5\text{x}10^{-5}\text{m})^2(600 - 1.4\text{ kgm}^{-3})}{18(1.8\text{x}10^{-5}\text{kgm}^{-1}\text{s}^{-1})} = 0.045\text{ m/s}$$

Checking the assumption of creeping flow for the seed, Re = $\rho_a w_P d/\mu_a$ = 0.15 < 1, so the creeping flow assumption is OK. If we assume that the release is instantaneous and occurs at a point, then we can write the following equation for the concentration downstream of the release. The settling velocity contributes an effective advection speed, w_P, such that the seed cloud's center of mass follows the trajectory (x = Ut, y = 0, z = H-w_Pt). A negative image is required to make the ground a perfect absorber, such that any seed that touches the ground settles and is removed from the air. The center of mass of the image source follows the trajectory (x = Ut, y = 0, z = -H+w_Pt). The first and second lines of the equation are the real and image source, respectively.

$$C(x,y,z,t) = \frac{N}{(4\pi t)^{3/2}\sqrt{D_x D_y D_z}} \exp\left(-\frac{(x-ut)^2}{4D_x t} - \frac{(y)^2}{4D_y t} - \frac{(z-(H-w_P t))^2}{4D_z t}\right)$$
$$- \frac{N}{(4\pi t)^{3/2}\sqrt{D_x D_y D_z}} \exp\left(-\frac{(x-ut)^2}{4D_x t} - \frac{(y)^2}{4D_y t} - \frac{(z-(-H+w_P t))^2}{4D_z t}\right)$$

This equation is valid at distances downstream of the tree for which 1) the travel time is long compared to the release time; and 2) the cloud's horizontal cross-section is large compared to the release area. The first condition is met at x >> $UT_{release}$= 10 m. For x >> 10 m, the concentration in the air appears as if released from an instantaneous source. The second condition is met when the horizontal size of the cloud is much greater than the cloud size at release. The tree area is 4m^2. Assuming a round tree, this area can be approximated as a circle of diameter 2 m. Thus the lateral dimension of the release is 2m. The longitudinal dimension of the release is set by the tree scale (2m) and by advection during release (10-m). The release is thus 2 m wide by 12 m long. Since $D_x = D_y$, the more restrictive condition will be on the longitudinal extent of the cloud. The release will appear as a point at distances for which $4\sigma_x = 4\sqrt{(2D_X x/U)}$ >> 12m, or x >> 9 m.

When the seeds begin to settle, two processes contribute to their longitudinal spread along the ground, 1) the longitudinal extent of the cloud as it settles ($4\sigma_x$) and 2) differences in settling time that result in differences in advected distance. The latter can be estimated by considering the vertical extent of the cloud ($4\sigma_z$). The resulting spread in settling times, $\Delta T = 4\sigma_z/w_s$, results in longitudinal spread of $\Delta x = U\,\Delta T$. We evaluate the relative contribution of these processes at the mean settling time (H/w_P).

$$\frac{\text{longitudinal diffusion}}{\text{differential settling and advection}} = \frac{\sigma_x}{U\,(\sigma_z/w_s)} = \frac{w_s\sqrt{2D_xH/w_s}}{U\sqrt{2D_zH/w_s}} = \frac{w_s}{U}\sqrt{\frac{D_x}{D_z}} = 0.07$$

That this ratio is so small indicates that the longitudinal distribution of seeds on the ground is predominantly determined by the differential settling times coupled to advection, and longitudinal diffusion contributes little additional spread. Now, to estimate the longitudinal distribution on the ground we need to know the time (and thus distance) at which the first and last seed settles. Considering the sketch below.

First seed settles when: $w_P t + 2\sqrt{2\,D_z t} = H$

Last seed settles when: $w_P t - 2\sqrt{2\,D_z t} = H$

Note that by using the 2σ contour to define the edge of the cloud we describe the fate of 95% of the seeds. Solving these two equations via a spreadsheet, the seeds are found to settle between t = 64 and 776 s. The seeds are thus distributed over the distances x = 128m to 1552 m, a spread of 1424 m. This is an overestimate, because in fact the velocity approaches zero near the ground due to the no-slip condition there. The neglected velocity shear would also contribute shear-dispersion that would augment longitudinally spreading.

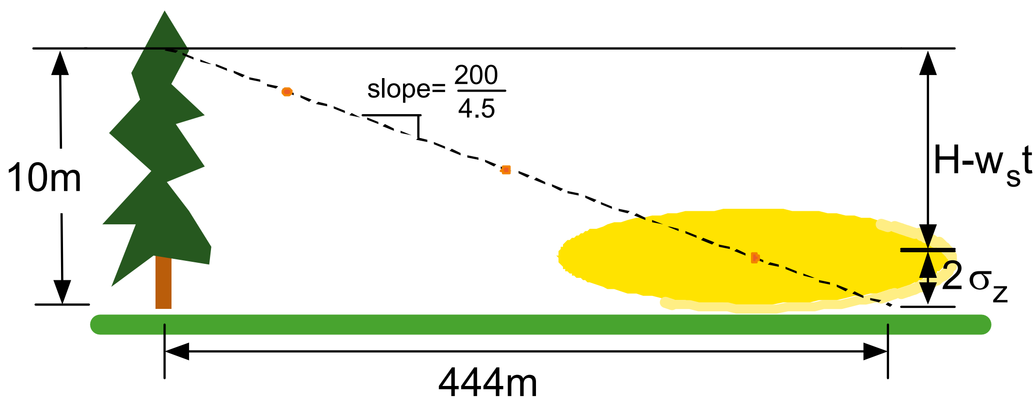


Figure is not to scale

Finally, for comparison, we estimate the longitudinal spread due only to longitudinal diffusion, i.e. neglecting. The extent of the cloud as the center of mass settles, *i.e.* at H/w_P, is $4\sqrt{(2\text{x}1\text{m}^2\text{s}^{-1}\text{x}222\text{s})} = 84$ m, which is much less than the spread accomplished by advection and differential settling (1.4 km), confirming the scaling analysis above.

Problem 10.4

Let the bubble rise velocity be w_B. A bubble requires the time $T_{rise} = h/w_B$ to reach the surface. In this time the bubble will travel a downstream distance of uT_{rise}. Thus the desired spacing scale is

(1) $L = uh/w_B$.

To find w_B we first assume creeping flow,

$$w_B = \frac{gd^2(\rho_B - \rho_W)}{18\mu_W} = \frac{9.8\text{ms}^{-2}(10^{-3}\text{ m})^2(1.4 - 998\text{ kgm}^{-3})}{18(10^{-3}\text{kgm}^{-1}\text{s}^{-1})} = 0.54\text{ms}^{-1}$$

Check $Re_B = (0.54\text{ms}^{-1})(0.001\text{m})/(10^{-6}\text{m}^2\text{s}^{-1}) = 540 >> 1$, so creeping flow is not confirmed. We now iterate to find the bubble velocity using the relations:

$$C_D = \frac{24}{Re_B} + \frac{3}{\sqrt{Re_B}} + 0.34 \quad \text{and} \quad w_B = \left[\frac{4}{3}\frac{gd\,(\rho_B - \rho_F)}{\rho_F\, C_D}\right]^{1/2}.$$

Assume $Re_B = 540$

$$C_D = \frac{24}{540} + \frac{3}{\sqrt{540}} + 0.34 = 0.51$$

$$w_B = \left[\frac{4}{3}\frac{(9.8\text{ms}^{-1})(0.001\text{m})\,(1.4 - 998\text{kgm}^{-3})}{(998\text{kgm}^{-3})(0.51)}\right]^{1/2} = 0.16\text{m/s}$$

$Re_B = (0.16\text{ms}^{-1})(0.001\text{m})/(10^{-6}\text{m}^2\text{s}^{-1}) = 160 \neq 540$, so go around again.

Assume $Re_B = 160$

$$C_D = \frac{24}{160} + \frac{3}{\sqrt{160}} + 0.34 = 0.73$$

$$w_B = \left[\frac{4}{3}\frac{(9.8\text{ms}^{-1})(0.001\text{m})\,(1.4 - 998\text{kgm}^{-3})}{(998\text{kgm}^{-3})(0.73)}\right]^{1/2} = 0.13\text{m/s}$$

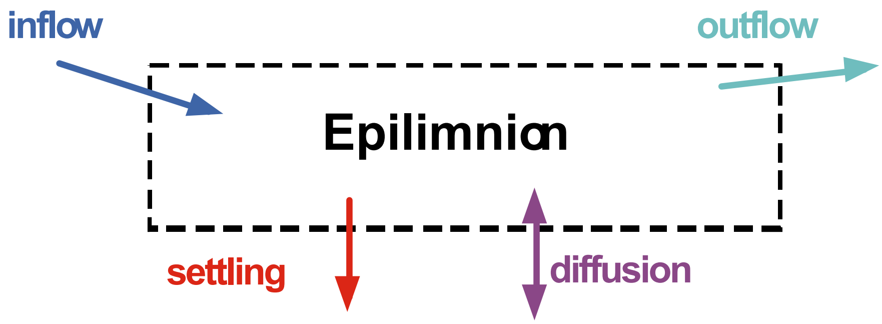
$Re_B = (0.13\text{ms}^{-1})(0.001\text{m})/(10^{-6}\text{m}^2\text{s}^{-1}) = 130 \approx 160$, we are close enough to stop.

Now, returning to (1),

$L = uh/w_B = (1\text{ms}^{-1})(1\text{m})/(0.13\text{ms}^{-1}) = 7.6$ m

Answer 10.5

To estimate whether phosphorus mass in the epilimnion will change in the near future, we consider a mass balance for this region. There are four fluxes contributing to and/or removing phosphorus from the epilimnion volume.



Inflow Flux = $Q_I C_I$ = (0.1 $m^3 s^{-1}$)(100µgL^{-1})(1000Lm^{-3}) = 10 mgs^{-1}.

Outflow Flux = $Q_O C_O$ = (0.1 $m^3 s^{-1}$)(50µgL^{-1})(1000Lm^{-3}) = 5 mgs^{-1}.

The settling flux occurs across the base of the epilimnion control volume. From the graph, this is located at 4m depth at which point C_{4m} = 40 µgL^{-1}. If we assume a cylindrical lake basin, the area at the base of the epilimnion is the same as the water surface area, A. Then,

Settling Flux = $w_P AC$ = (2.0 x 10^{-6} ms^{-1})(2 x 10^4 m^2)(40µgL^{-1})(1000Lm^{-3}) = 1.6 mgs^{-1}.

The diffusive flux also occurs across the base of the epiliminion. We need to estimate the gradient of concentration at the base. For this we can use a central difference z = 4m,

$(\partial C/\partial z)_{4m} = (C_{3m} - C_{5m})/(3 - 5\text{ m})$ = (50-30 µgL^{-1})/(3-5m) = -10µg$L^{-1}m^{-1}$.

Diffusive Flux =

$- D_t A\, \partial C/\partial z$ = -(2x10^{-6} $m^2 s^{-1}$)(2x10^4 m^2)(-10µg$L^{-1}m^{-1}$)(1000Lm^{-3}) = + 0.4 mgs^{-1}.

The positive sign indicates that the diffusive flux is directed downward, along the positive z axis. This flux is a loss for the control volume.

The sum of these fluxes gives the rate of change of phosphorus mass in the epilimnion,

$\partial M/\partial t$ = + Inflow - Outflow - Settling - Diffusive Flux = (+10 - 5 - 1.6 - 0.4) = +3mgs^{-1}.

With $\partial M/\partial t > 0$, the phosphorus levels in the epilimnion will continue to increase.