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1.061 / 1.61 Transport Processes in the Environment  
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## 10. Transport of particles



This chapter provides an introduction to the transport of particles that are either more dense (e.g. mineral sediment) or less dense (e.g. bubbles) than the fluid. A method of estimating the settling velocity of particles is explained, and then the loss of settling particles from a laminar flow and from a turbulent flow are contrasted. A simple scaling analysis tells us that if the settling velocity of the particles is much less than the friction velocity of the flow, then the turbulence will be sufficiently vigorous to keep the particles in suspension.

Sample problems require the user to gauge the settling (or rise) velocity of suspended particles and determine the effect that it has on downstream particle concentration.

## 10. Introduction to the Transport of Particles

Small, neutrally buoyant particles exactly follow the fluid flow,  $(u, v, w)$ , such that their transport is described by the same equation used for dissolved chemicals. Particles whose density deviates from that of the fluid, either more (e.g. mineral grains) or less (e.g. gas bubbles), will have a vertical velocity relative to the fluid,  $w_p$ , which constitutes an additional component of advection for the particle. The particle velocity is proportional to the density difference between the particle and the fluid and to the particle diameter,  $d$ . With  $z$  taken as the vertical coordinate, the additional velocity component appears in the third term of advective flux.

$$(1a) \quad \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + (w + w_p) \frac{\partial C}{\partial z} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial y^2} + D_z \frac{\partial^2 C}{\partial z^2}$$

In (1a), we have assumed that the particles have a narrow range of size and density, such that  $w_p$  is the same for all particles in the flow. If the range of particle size and density is large, then multiple transport equations must be evaluated, each for a different sub-set of particles with comparable  $w_p$ . In (1a) the concentration is given in  $C$  [ $\text{kg m}^{-3}$ ]. In some cases it is more convenient to consider the particle concentration as  $n$  [particles  $\text{m}^{-3}$ ]. If the distribution of particle diameter,  $d$ , and density,  $\rho_p$ , is narrow, then  $C = \rho_p(\pi/6)d^3 n$ . Dividing through by  $\rho_p(\pi/6)d^3$ , (1a) becomes,

$$(1b) \quad \frac{\partial n}{\partial t} + u \frac{\partial n}{\partial x} + v \frac{\partial n}{\partial y} + (w + w_p) \frac{\partial n}{\partial z} = D_x \frac{\partial^2 n}{\partial x^2} + D_y \frac{\partial^2 n}{\partial y^2} + D_z \frac{\partial^2 n}{\partial z^2}.$$

In turbulent flow, the diffusion coefficients for particles can be assumed to be the same as those for dissolved species. Under laminar flow conditions, particle diffusion will be a function of particle size, as shown below.

### Particle Velocity

To determine the particle velocity, we apply conservation of vertical momentum to a particle considering the forces of weight, buoyancy and drag. For simplicity, we consider a solid, spherical particle of diameter,  $d$ , and density,  $\rho_p$ . The fluid density is  $\rho_F$ . In a coordinate system with  $z$  positive upward, we have

$$\text{Weight} = -\rho_p g (\pi/6) d^3$$

$$\text{Buoyancy} = \rho_F g (\pi/6) d^3$$

$$\text{Drag} = - (1/2) \rho_F C_D (\pi/4) d^2 w_p |w_p| \quad [\text{sign of drag will be opposite to velocity}]$$

$C_D$ , the drag coefficient for a sphere, depends on the Reynolds number defined by the particle velocity, i.e.  $Re_p = d w_p / \nu$ . The conservation of vertical momentum is then,

$$(2) \quad \rho_p (\pi/6) d^3 \frac{\partial w_p}{\partial t} = (\rho_F - \rho_p) g (\pi/6) d^3 - \frac{1}{2} \rho_F C_D (\pi/4) d^2 w_p |w_p|$$

Consider a particle starting from rest ( $w_p = 0$ ). If  $\rho_p > \rho_F$  the particle begins to accelerate downward ( $\partial w_p / \partial t < 0$ ). As  $w_p$  increases, the drag on the particle increases, and acts in the opposite direction to  $w_p$ . Eventually the drag grows large enough to exactly balance the particle weight and buoyancy, making the right-hand side of (2) zero. At this point the particle acceleration becomes zero and  $w_p$  becomes constant. This is called the terminal velocity. Typically the time to reach the terminal velocity is very short compared to the time scale of interest, so that terminal velocity is assumed for all time. When terminal velocity is reached,  $\partial w_p / \partial t = 0$ , and (2) can be solved for  $w_p$ .

$$(3) \quad |w_p| = \left[ \frac{4}{3} \frac{gd(\rho_p - \rho_F)}{\rho_F C_D} \right]^{1/2}$$

where  $C_D = f(\text{Re}_p)$  is described by the following empirical approximation for  $\text{Re}_p < 10^4$ .

$$(4) \quad C_D = \frac{24}{\text{Re}_p} + \frac{3}{\sqrt{\text{Re}_p}} + 0.34 \quad \text{for } \text{Re}_p < 10^4$$

For  $\text{Re} < 1$  (called creeping flow), an analytical solution exists for the drag, which yields:

$$(5) \quad C_D = \frac{24}{\text{Re}_p} \quad \text{for } \text{Re} < 1 \text{ [creeping flow]}$$

Using (5), we can simplify (3)

$$(6) \quad w_p = \frac{gd^2(\rho_p - \rho_F)}{18\mu_F} \quad \text{for } \text{Re} < 1 \text{ [creeping flow]} \quad \mu_F = \rho_F \nu_F$$

(6) is called the Stokes velocity. Because  $\text{Re}_p$  depends on  $w_p$ , we do not know *a priori* whether (6) will apply. One can assume creeping flow, find  $w_p$  using (6), and then confirm the assumption of creeping flow. If creeping flow is not confirmed, then an iterative solution is needed (see example 2 below)

*Example 1: Find the settling (particle) velocity for 0.01mm diameter quartz sand.*

The density of quartz is  $2600 \text{ kgm}^{-3}$ . The density and kinematic viscosity ( $\nu$ ) of water are  $\approx 1000 \text{ kgm}^{-3}$  and  $10^{-6} \text{ m}^2 \text{ s}^{-1}$ . Assume  $\text{Re}_p < 1$ , then

$$w_p = (9.8 \text{ ms}^{-2}) (10^{-5})^2 (2600 - 1000 \text{ kgm}^{-3}) / (18 \times 1000 \text{ kgm}^{-3} \times 10^{-6} \text{ m}^2 \text{ s}^{-1}) = 9 \times 10^{-5} \text{ ms}^{-1}.$$

Check assumption of creeping flow

$$\text{Re}_p = w_p d / \nu = (9 \times 10^{-5} \text{ ms}^{-1}) (10^{-5} \text{ m}) / (10^{-6} \text{ m}^2 \text{ s}^{-1}) = 9 \times 10^{-4} \ll 1 \quad \text{check}$$

*Example 2: Find the settling (particle) velocity for 1mm diameter quartz sand.*

Assume  $Re_p < 1$ , then from (6)

$$w_p = (9.8 \text{ ms}^{-2}) (10^{-3})^2 (2600-1000 \text{ kgm}^{-3}) / (18 \times 1000 \text{ kgm}^{-3} \times 10^{-6} \text{ m}^2 \text{ s}^{-1}) = 9 \times 10^{-1} \text{ ms}^{-1}.$$

Check assumption of creeping flow

$$Re_p = w_p d / \nu = (9 \times 10^{-1} \text{ ms}^{-1})(10^{-3} \text{ m}) / (10^{-6} \text{ m}^2 \text{ s}^{-1}) = 900 \gg 1 \quad \text{not creeping flow}$$

Use estimated  $Re_p = 900$  to estimate  $C_D$

$$\text{From (4) } C_D (Re_p = 900) = 0.47$$

$$\text{Then from (4) } w_p = 0.2 \text{ ms}^{-1}$$

$$\text{New } Re_p = (0.001 \text{ m})(0.2 \text{ ms}^{-1}) / 10^{-6} \text{ m}^2 \text{ s}^{-1} = 200$$

Guess of  $Re_p = 900$  does not match resulting  $Re_p = 200$ . Use new  $Re_p$  to repeat process.

Guess  $Re_p = 200$ ; then  $C_D = 0.67$ ; and  $w_p = 0.17 \text{ ms}^{-1}$ ; yielding  $Re_p = 170$ .

Guess of  $Re_p = 200$  does not match resulting  $Re_p = 170$ . Repeat once more.

**Guess  $Re_p = 170$** ; then  $C_D = 0.71$ ; and  $w_p = 0.17 \text{ ms}^{-1}$ ; **yielding  $Re_p = 170$**

When the resulting  $Re_p$  matches the guessed  $Re_p$  within 10%, you can stop.

### *Effects of Particle Shape*

Most quartz grains (common beach sand) are roughly spherical and solid, such that (6) and (3) work well. Bubbles also fit the assumptions of (6) and (3) very well. However, many mineral grains and clays have flat or flake-like structure, *i.e.* not spherical. These particles do not fall straight down, but tend to waft in a zig-zag pattern, like a leaf falling. So (3) and (6) may only be taken as ball-park values, with the deviation in velocity between flake and spherical morphology increasing as  $Re_p$  increases. Finally, many particles are not solid, but are aggregates (flocs) of smaller particles, which can be quite porous. When flocs are very porous, their effective density is reduced to a value closer to the water, and  $w_p$  is decreased.

### **Particle Diffusion**

If the fluid flow is laminar, the diffusion of particles, like the diffusion of dissolved molecules, depends on the Brownian motion of the fluid molecules. The particle diffusion coefficient,  $D_p$ , is described by the

$$\text{Stokes-Einstein Equation:} \quad D_p = \frac{kT}{6\pi\mu r}, \quad (7)$$

where  $r$  is the particle radius,  $T$  [°K] is the absolute temperature, and  $k$  is the Boltzmann constant,  $k = 1.381 \times 10^{-23} \text{ J/}^\circ\text{K}$ . This equation is based on a random walk model, in which each step executed by a suspended particle is caused by the impact of a fluid molecule. The impact transfers kinetic energy from the fluid molecule to the particle, such that immediately following the impact the particle has kinetic energy,

$$(1/2)m u_o^2 = (1/2) k T, \quad (8)$$

where  $u_o$  is the initial velocity of the particle after the collision and  $m$  is the particle mass. The subsequent motion of the particle is described by the momentum equation,

$$\frac{d}{dt}(mu) = -\frac{1}{2}\rho_F C_D \pi r^2 u^2. \quad (9)$$

Assuming that  $Re_p < 1$ ,  $C_D = 24/Re_p$  and (9) becomes

$$m \frac{du}{dt} = -6\pi\mu r u \quad (10)$$

From which the particle motion can be represented as

$$u(t) = u_o \exp(-t/\tau), \quad (11)$$

where  $\tau = m/(6\pi\mu r)$  is a time-scale describing the duration of motion before the particle returns to rest relative to the mean flow. Using this time scale and the initial velocity,  $u_o$ , the distance traveled after one collision is

$$\ell = u_o \tau = \frac{\sqrt{mkT}}{6\pi\mu r}. \quad (12)$$

A suspended particle experiences a continuous sequence of collisions, and after each collision it takes time  $\tau$  to move a distance  $\ell$  along the line of impact. The net result is a random walk with step size  $\Delta x = \ell$  and step time  $\Delta t = \tau$  that results in a Fickian Diffusion. The coefficient of diffusion, as defined in [Chapter 1](#), is

$$D_p = \frac{\Delta x^2}{\Delta t} = \frac{\ell^2}{\tau} = \frac{kT}{6\pi\mu r}.$$

### Instantaneous Point Source of Particles

In previous chapters we derived solutions for the concentration field created by point sources. With the above considerations, these solutions can be used to describe particle concentration as well. As an example we consider a cloud of  $N$  particles released from a height,  $z = h$ , and  $x = y = 0$  into a domain unbounded in  $x$  and  $y$ . The cloud is advected by a mean velocity,  $u$ , and diffused by an isotropic turbulent diffusivity,  $D$ . We will assume that any particle that touches the ground ( $z = 0$ ) settles out and cannot be resuspended, so the ground is a perfect absorber. The center of mass for the cloud will be at  $(x = ut, y = 0, z = h - w_p t)$ , and for the negative image at  $(x = ut, y = 0, z = -h + w_p t)$ . The particle concentration,  $n$  [particles  $m^{-3}$ ], is

$$\begin{aligned}
 n = & \frac{N}{(4\pi Dt)^{3/2}} \exp\left(-\frac{(x-ut)^2 + y^2 + (z-(h-w_p t))^2}{4Dt}\right) \\
 & - \frac{N}{(4\pi Dt)^{3/2}} \exp\left(-\frac{(x-ut)^2 + y^2 + (z+(h-w_p t))^2}{4Dt}\right)
 \end{aligned}
 \tag{13}$$

### Settling - A sink for suspended particle concentration.

As in the above example, the settling of particles onto a boundary represents a flux of particles out of the fluid domain (a sink). With the positive  $z$ -axis pointing upward, the flux at the boundary is  $\dot{m} = -w_p C A_H$ , where  $A_H$  is the horizontal projection of the boundary and  $C$  is the concentration in the fluid next to the boundary. In some systems, particles that have settled can be resuspended. Resuspension of particles creates a flux into the fluid domain (a source). The ability of a flow to resuspend particles from the bed depends on the shear stress exerted at the boundary and the physical characteristics of the particles. The relative magnitude of the settling and resuspension determines whether there is a net loss or gain particles from the fluid. In this chapter we ignore resuspension and consider only the settling flux.

#### *Settling in a System with Laminar Flow or Slow Mixing*

Consider a simple rectangular system with horizontal area  $A$  and depth  $h$  and  $(u, v, w) = (0, 0, 0)$ . Let  $z$  be vertically upward and  $z = 0$  at the bottom. The system has an initial concentration  $C_0$  that is uniform throughout the fluid domain. If diffusion is slow compared to settling, then we can neglect diffusion (mixing), and assume that the concentration within the particle cloud is unchanged as settling progresses. Assuming a cloud of uniform particle size and density, the particles will all settle at the same velocity  $w_p$ . The particle flux at the bed is due to vertical advection,  $\dot{m}(z=0) = -w_p C_0 A$ . This flux continues until the entire water depth is cleared of particles. Under these conditions the loss of particle mass,  $M$ , follows zeroth-order decay.

$$(14) \quad \partial M / \partial t = -w_p C_0 A = \text{constant and not a function of mass remaining, } M.$$

If we define a depth-averaged concentration as  $C = M/hA$ ,

$$(15) \quad \partial C / \partial t = - (w_p / h) C_0,$$

From which we get,

$$(16) \quad C(t) = C_0 \left(1 - \frac{w_p}{h} t\right), \text{ for } t < h/w_p.$$

All particles are lost from the system in exactly  $T_{\text{settle}} = h / w_p$ .

*Settling in a System with Turbulent Flow or Rapid Mixing*

For the same system described above now suppose that mixing is sufficiently rapid to maintain a uniform concentration,  $C$ , throughout the system, even as particles are lost to the bed through settling. The flux at the bed is now  $m(z=0) = -w_p CA$ . Although diffusion (mixing) cannot be neglected in this system, we have assumed that  $C$  is uniform, so we may neglect the diffusion terms because  $\partial C/\partial z = \partial C/\partial y = \partial C/\partial x = 0$ . The mass conservation equation for this system is,

$$(17) \quad \partial M/\partial t = Ah \partial C/\partial t = -w_p CA,$$

from which

$$(18) \quad \partial C/\partial t = -(w_p/h) C,$$

which is a first-order decay, with a rate constant  $k[\text{time}^{-1}] = w_p/h$ . The particle concentration decays exponentially, with 95% of the initial mass lost in time  $3h/w_p$ .

*Choosing a settling model*

The two models described above differ in the relative importance of mixing and settling. We compare these two processes by comparing the time-scale for settling over the depth,  $T_{\text{settle}} \sim h/w_p$ , and the time scale for mixing over the depth,  $T_D \sim h^2/D$ . If  $T_D \gg T_{\text{settle}}$ , the slow-mixing model will apply. If  $T_D \ll T_{\text{settle}}$ , the fast-mixing model will apply.

Using the scale  $D \sim u_* h$ , for turbulent channel flow, we find the ratio of time-scales,

$$(19) \quad \frac{\text{time - scale for settling over } h}{\text{time - scale for mixing over } h} = \frac{h/w_p}{h^2/D} = \frac{w_p}{u_*}.$$

Then, for turbulent channel flow, if  $w_p \ll u_*$ , the turbulence in the water column is strong enough to keep the particles mixed, the fast-mixing model applies, and the suspended sediment load decays exponentially. If  $w_p \gg u_*$ , the turbulence is too weak to mix sediment vertically, the slow-mixing model applies, and the suspended sediment load decays linearly.