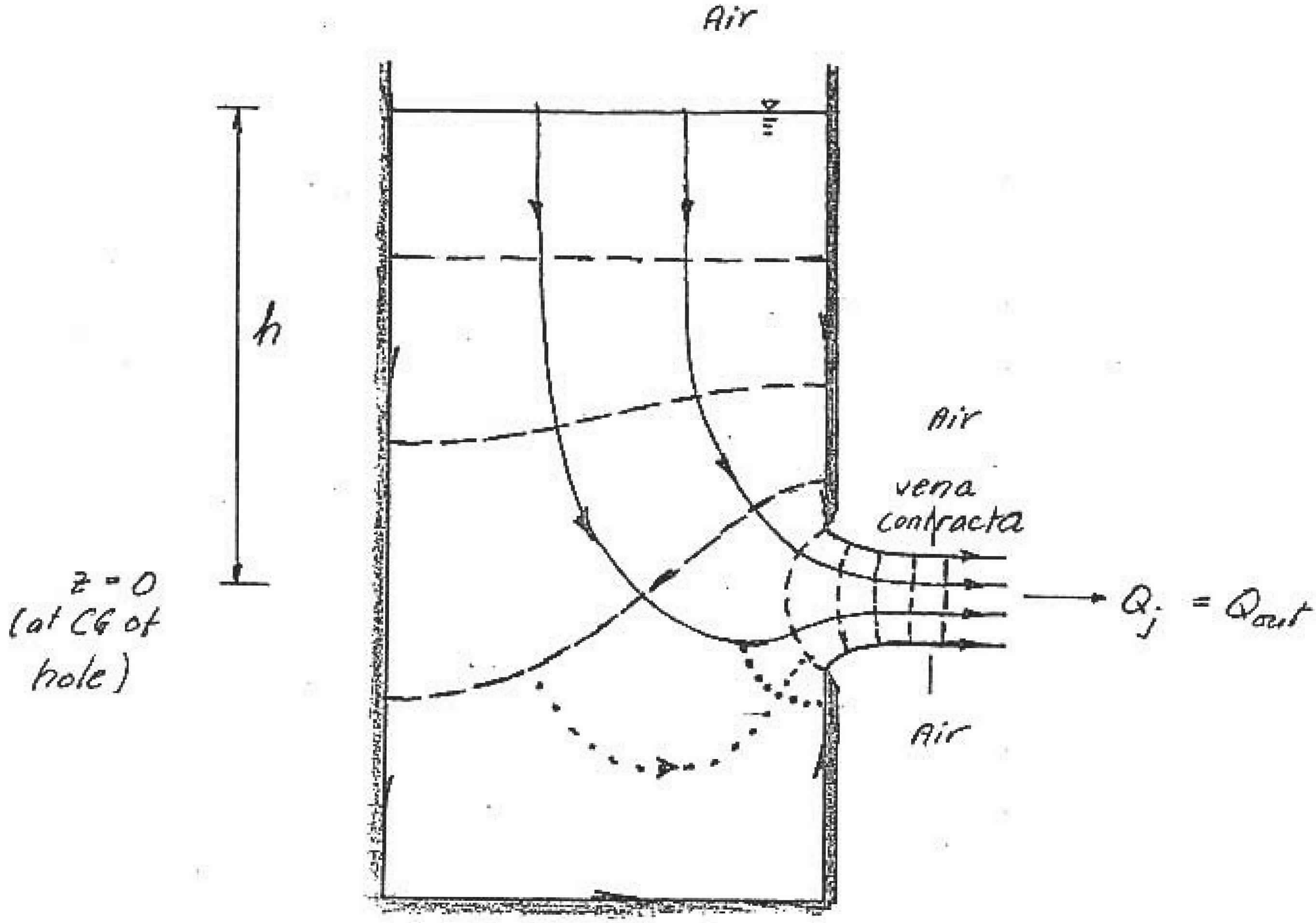
RECITATION #4

1.060 ENGINEERING MECHANICS II

THERE'S A HOLE IN THE BUCKET

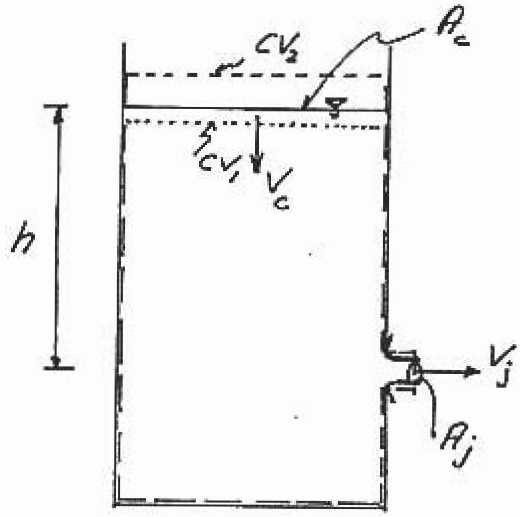
The Problem



Large container bucket : Surface Area A_c
Small hole in bucket : Hole Area A_h
Vena contracta jet : Area = $A_j = C_v A_h$

How fast does water exit the bucket, or what is Q_j as the bucket is emptied?

1 Conservation of Volume (Continuity)



CV_1 has top surface just below free surface in bucket

CV_2 has top surface just above free surface in bucket

Otherwise, $CV_1 = CV_2$

CV_1 is filled with water at all times. Conservation of volume (of water) within CV_1 (it is constant) gives

$$\text{In} - \text{Out} = Q_{in} - Q_{out} = V_c A_c - V_j A_j = \text{change} = 0$$

$$V_c = (A_j / A_c) V_j \tag{1.1}$$

CV_2 does not have an inflow (of water), but volume of water inside CV_1 changes with time. Conservation of volume gives

$$\text{In} - \text{Out} = -Q_{out} = -V_j A_j = \frac{d}{dt}\Big(h A_c + \overset{\text{constant}}{\overset{''}{V_0}}\Big) = A_c \frac{dh}{dt}$$

$$-\frac{dh}{dt} = (A_j / A_c) V_j \tag{1.2}$$

Since $(-dh/dt)$ = velocity at which free surface is moving down, this must equal V_c, and (1.1) and (1.2) are one and the same!

2. Estimate of V_j, the Outflow Velocity

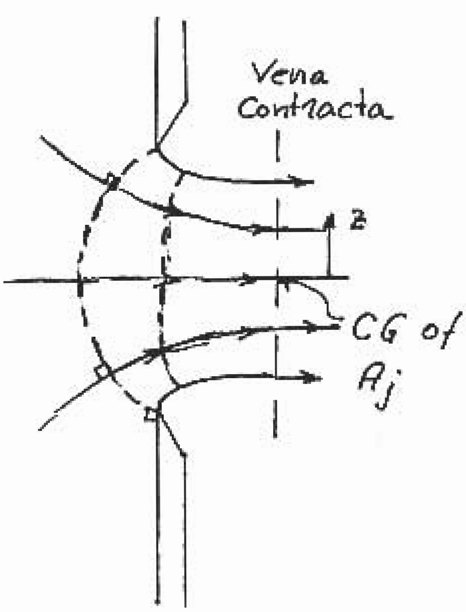
All streamlines must start at the free surface in the bucket and they must all pass the area, A_j, of the water jet shooting out of the hole in the bucket.

Along any streamline we have therefore from Bernoulli's equation [assuming flow to be steady]

$$\tfrac{1}{2}\rho V_j^2 + p_j + \rho g z_j = \tfrac{1}{2}\rho V_c^2 + p_c + \rho g z_c \qquad (2.1)$$

From (1.1) we have $V_c = (A_j/A_c)V_j$, since streamline starts at free surface $p_c = p_{atm} = 0$, and choosing $z=0$ at the center of gravity of A_j we have $z_c = h$. With this information, (2.1) may be written

$$V_j^2\left(1-(A_j/A_c)^2\right) = 2g(h-z_j) - 2p_j/\rho \qquad (2.2)$$



In vicinity of the hole streamlines are curved, so pressure is <u>not</u> hydrostatic across the area of the hole in the bucket wall. However, after the flow has turned the corner the streamlines become straight (and parallel) at the so-called "vena contracta". Our control volume is choosen to cut across the jet at its "vena contracta"

First, these considerations suggest that

$$A_j = C_v A_h \tag{2.3}$$

in which A_h = area of the hole in the buckets wall, and C_v is a contraction coefficient. For circular holes $C_v \simeq 0.6$, which is also the value for "plane" flow out of a long rectangular orifice.

Second observation from conditions at "veno contracta" is that the pressure is atmospheric all around the jet. With streamlines being straight line, there is no dynamic pressure (V^2/R) inside the jet, so for any point on A_j

$$p_j \simeq 0 \tag{2.4}$$

Introducing these results in (2.2) gives

$$V_j^2\left(1-\left(C_v A_h / A_c\right)^2\right) = 2g\left(h - z_j\right) \tag{2.5}$$

in which z_j is the value of z for a particular streamline, as it passes A_j (vena contracta).

From (2.5) we finally obtain the velocity distribution across vena contracta

$$V_j = \sqrt{2gh}\,\frac{\sqrt{1 - z_j/h}}{\sqrt{1-\left(C_v A_h/A_c\right)^2}} \tag{2.6}$$

First, we notice that accounting of the velocity, V_{c}, at the free surface of the bucket when specifying the Bernoulli constant has resulted in the factor

$$\sqrt{1-(C_v A_h / A_c)^2} \approx 1 \qquad (2.7)$$

for a hole that is reasonable small relative to the area of the bucket, e.g. $A_h / A_c = 0.15$ [LARGE HOLE!] and $C_v = 0.6$ gives the factor 0.996, which for all practical purposes is equal to unity.

Second, in order to obtain the discharge out of the bucket, we should really obtain this by integration over vena contracta, i.e.

$$Q_j = \int_{A_j} V_j \, dA = \sqrt{2gh} \int_{A_j} \sqrt{1 - z_j/h} \, dA \qquad (2.8)$$

The integral is equal to A_j if we assume $z_j/h \ll 1$, but we can do better than that since for z_j/h reasonably small

$$\sqrt{1 - z_j/h} \simeq 1 - \tfrac{1}{2} z_j/h$$

(e.g. $z_j/h = 0.3$ gives $\sqrt{0.7} = 0.84$ whereas the approximation gives 0.85. Not bad). With this approximation, the integral in (2.8) may be written

$$Q_j \simeq \sqrt{2gh}\left(A_j - \frac{1}{2h}\int_{A_j} z_j\, dA\right) = \sqrt{2gh}\, A_j \qquad (2.9)$$

by definition of the center of gravity of A_j, and having chosen this as our origin for z!

Equation (2.9) shows that the expression used for the outflow in the application of the continuity principle,

$$Q_j = V_j A_j = \sqrt{2gh}\, A_j$$

is an excellent approximation provided that we take

$$V_j = \sqrt{2gh} \qquad (2.10)$$

with h being the vertical distance to the free surface in the bucket measured from the center of gravity of the hole in the bucket.

3. Effect of Unsteadiness

Introducing (2.10) in (1.2) we obtain an equation for the variation of surface level in the bucket as a result of the hole. This equation is

$$\frac{dh}{dt} = -\left(A_j/A_c\right)V_j \simeq -\frac{C_v A_h}{A_c}\sqrt{2gh} = -K h^{1/2} \qquad (3.1)$$

with K being a "constant" (with dimensions).

This equation is readily solved to give

$$\int_{h_0}^{h} h^{-\frac{1}{2}} dh = 2(\sqrt{h} - \sqrt{h_0}) = -\int_0^t K dt = -Kt$$

or

$$\sqrt{h} = \sqrt{h_0} - \frac{K}{2} t = \sqrt{h_0} - \frac{C_v A_h}{\sqrt{2} A_c} \sqrt{g}\, t \qquad (3.2)$$

which may be used to estimate the length of time required to empty the bucket from its initial level, $h = h_0$ at $t = 0$, to a lower level h. However, as h approaches zero, we must recall that our solution assumed, among other things, $z_j / h \ll 1$ which is surely violated as $h \to 0$!

Equations (3.1) and (3.2) show that the flow we have considered based on the Bernoulli Equation for steady flow in fact is unsteady! Now, for an unsteady flow the Bernoulli equation is obtained from integration along a streamline

$$\int_{s_1}^{s_2} \frac{\partial}{\partial s}\left[\frac{1}{2}\rho V_s^2 + p_s + \rho g z_s\right] ds = -\int_{s_1}^{s_2} \rho \frac{\partial V_s}{\partial t} ds \qquad (3.3)$$

or, for our problem, with $s_1 = s_c = 0$ at the surface in the bucket and $s_2 = s_j$ at vena contracta, we have

$$V_j^2\left(1 - (A_j / A_c)^2\right) = 2gh\left(1 - \frac{z_j}{h}\right) - 2\int_{s_c}^{s_j} (\partial V_s / \partial t)\, ds \qquad (3.4)$$

Without unsteady effects we obtained

$$V_j \approx \sqrt{2gh}$$

so

$$\partial V_j/\partial t = \frac{1}{2}\sqrt{2g}\, h^{-\frac{1}{2}} \frac{\partial h}{\partial t} = \sqrt{2gh}\,\frac{-V_c}{2h} = -2gh\left(\frac{A_j}{A_c}\right)\frac{1}{2h} \qquad (3.5)$$

represents the maximum acceleration ($V_j \gg V_s$) along any streamline from the surface in the bucket to vena contracta. At the start of the streamline, at the free surface in the bucket, we have

$$\frac{\partial V_c}{\partial t} = \left(A_j/A_c\right)\frac{\partial V_j}{\partial t} = -2gh\left(\frac{A_j}{A_c}\right)^2\frac{1}{2h} \qquad (3.6)$$

which represents a minimum value of $\partial V_s/\partial t$.

By (3.5) and (3.6), with the length of any streamline $S_j - S_c \approx h$, we obtain from (3.4)

$$V_j^2\left(1-(A_j/A_c)^2\right) = 2gh\left[\left(1-\frac{z_j}{h}\right) + \begin{Bmatrix} A_j/A_c \\ (A_j/A_c)^2 \end{Bmatrix}\right] \qquad (3.7)$$

which shows the effect of unsteady flow to increase the predicted velocity at vena contracta by a factor, F_u, bounded by

$$\sqrt{1+(A_j/A_c)^2} < F_u < \sqrt{1+A_j/A_c} \qquad (3.8)$$

with F_u most likely closer to the lower than to the higher bound.

4. General Conclusions

For converging flows from a large crossection to a small crossection one may

- Neglect the velocity in the large crossection when evaluating the Bernoulli constant.

- Choose origin of z at center of gravity, CG, of small crossectional area A_{small}

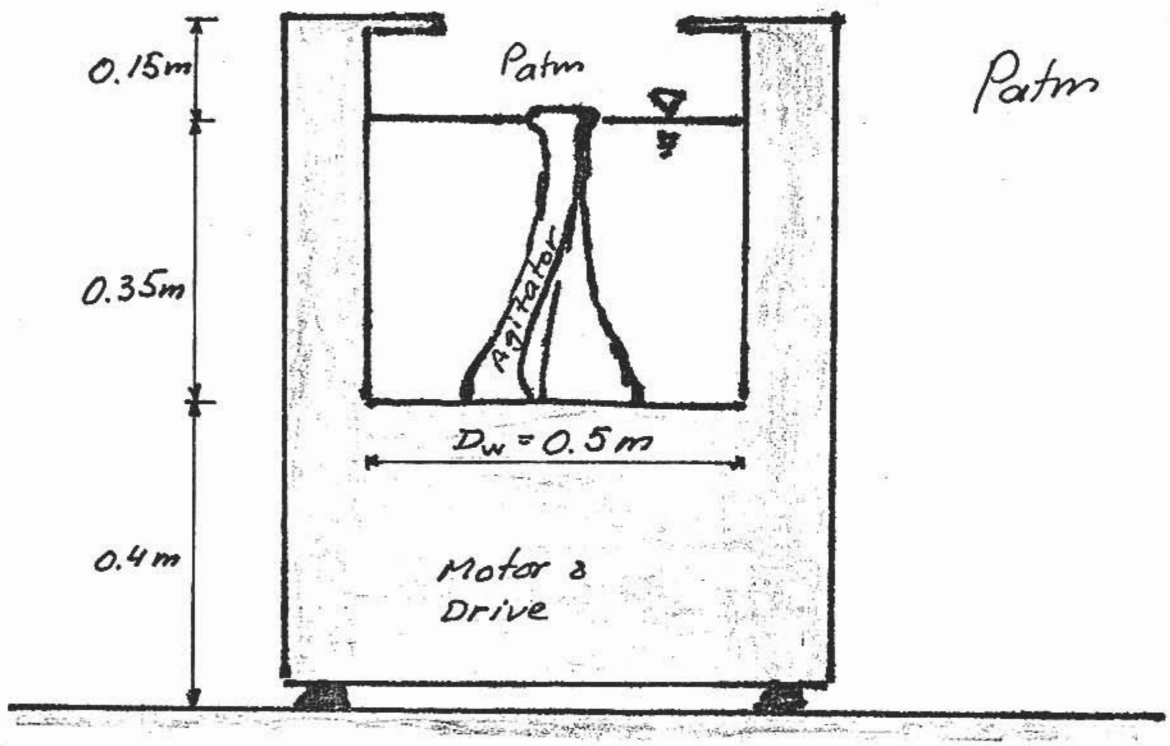
- Evaluate velocity from streamline passing through CG of A_{small}, and obtain discharge by multiplying this velocity and A_{small}.

- Neglect unsteady effects in Bernoulli's equation.

In all cases, when in doubt, check assumptions' validity. This note should provide you with a useful example of how one may proceed to check the validity of the above assumptions.

RECITATION #4

Problem No: 1



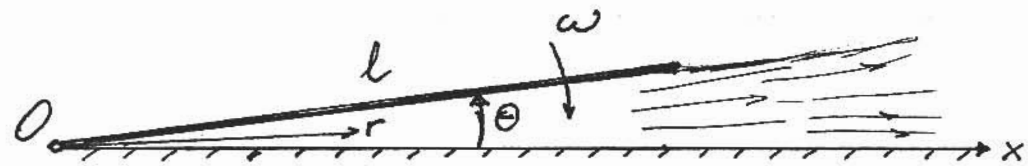
Your washer breaks down just after completing the hot washing cycle. You need to get rid of the hot water in order to examine the situation and find out what went/is wrong with the washer.

You have a 10ft long, 1/2 inch (diameter) 10ft long rubber hose, a standard bucket, and a floor drain located 3 ft. from the washer.

How are you going to get rid of the hot water, and how long will it take? (treat water as incompressible & inviscid)

Problem No: 2

Palm



A plate of length l (very long into paper, so problem may be treated as 2-D) rotates around a pivot point, O, at an angular velocity ω [rad/s] forcing the plate down towards a horizontal surface. The angle between the rotating plate and the surface, θ, is decreasing with time and water filling the gap is being squeezed out. $\theta \ll 1$ [rad] so $\sin\theta = \tan\theta \approx \theta$, $\cos\theta \approx 1$.
Treat fluid as incompressible and inviscid.

1) Determine the velocity in the gap as a function of radial distance r from O, (since $\cos\theta \approx 1$ you call it x or r, up to you), θ and ω.

2) Assuming steady flow and neglecting elevation differences (take $z = 0$ everywhere) determine the pressure variation along the plate. $p(r)$

(over)

3) Integrate $p(r)$ obtained in (2) along the length of the plate ℓ to obtain the total pressure force (per unit length into paper), $P_{(p)}$, acting on the plate, its direction and vertical component.

4) For $\ell = 8\,cm$, $\omega = 100\ rad/s$, $\rho = 1{,}000\ kg/m^3$, evaluate the force $P_{(p)}$ obtained in (3) for $\theta = 15°$ and $7°$.

5) If you were to account for unsteadiness would you expect this to be important? and would it increase or decrease your estimate of P_p?

6) If you were to account for frictional forces along plate and horizontal surface, would P_p increase or decrease?

RECITATION #4

Problem NO: 1

1) Submerge the rubber hose in the hot water in the washing machine so that it is completely full of water.

2) Close one end of rubber hose with your thumb and lift it out of washing machine and locate it above the floor drain. During this operation, the other end of the hose should remain submerged in the washing machine.

You have primed the siphon - let go with the thumb and water will start running.

Velocity in hose

Use Bernoulli from washing machine to floor drain

H_w = head in washer = $z_w + P_w/\rho g + V_w^2/2g$ = $0.4 + h + 0$ = head at inflow to hose

($z = 0$ at floor level, h = depth of water in washer, V_w = velocity of water in washer $\approx$ 0 because of the large flow area).

H_{fd} = head at floor drain = $\underset{0}{P_{atm}/\rho g} + \underset{0}{z_{fd}} + \frac{V_h^2}{2g}$

V_h = velocity in hose = $\sqrt{2g(h+0.4)}$ (SI)

Discharge

$$Q_h = \text{discharge in hose} = V_h \cdot \frac{\pi}{4} d_h^{\,2} =$$

$$\sqrt{2g(h+0.4)} \cdot \frac{\pi}{4}(0.0127)^2 = 5.6 \cdot 10^{-4} \sqrt{h+0.4} \; (SI)$$

Conservation of Volume

$$-\frac{\partial h}{\partial t}\left(\frac{\pi}{4} D_w^{\,2} - \underbrace{\text{area of agitator}}_{\approx 0}\right) = Q_h$$

or

$$\frac{dh}{dt} = \frac{Q_h}{\frac{\pi}{4} \cdot 0.5^2} = -2.9 \cdot 10^{-3} \sqrt{h+0.4}$$

or

$$\frac{dh/dt}{\sqrt{h+0.4}} = -2.9 \cdot 10^{-3} \Rightarrow \frac{dh}{\sqrt{h+0.4}} = -2.9 \cdot 10^{-3} dt$$

Integrate from $t=0$ $(h=0.35m)$ to t $(h=h)$ to get

$$\left[2\sqrt{h+0.4}\right]_{0.35}^{h} = 2\left(\sqrt{h+0.4} - \sqrt{0.35+0.4}\right) =$$

$$2\left(\sqrt{h+0.4} - \sqrt{0.75}\right) = -2.9 \cdot 10^{-3}\, t$$

$h = 0$ – washing machine is empty – and

$$\underline{t_{empty}} \cong \frac{2\left(\sqrt{0.75} - \sqrt{0.4}\right)}{2.9 \cdot 10^{-3}} = 161\,\text{sec} \approx \underline{2 \text{ to } 3\,\text{min}}$$

RECITATION #4

Problem No: 2

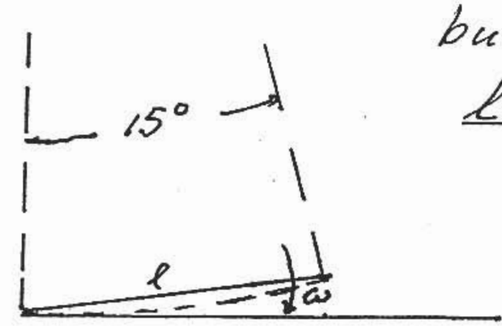
$r = 30$ cm $\approx$ radius of a car's wheel

$V_c = 65$ mph $= 29$ m/s $=$ speed of the car

$V_c \cdot T = 2\pi r \Rightarrow \underline{\omega} =$ rad. freq. of wheel $= V_c / r \approx \underline{100 \frac{rad}{s}}$

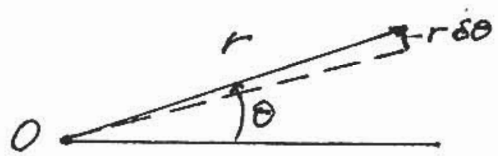
Circular wheel approximated by 24 sided polygon $- \; 360/24 = 15^\circ = 0.26$ rad. $\Rightarrow$ Bumpy ride, but simpler to analyse.

$\underline{\ell} \cong r \sin 15^\circ = \underline{7.8 \text{ cm}}$



Our problem simulates a tire hitting the road when it is covered by a layer of water.

Conservation of Volume



$$Q_{in} - Q_{out} = 0 - V(r)\, r\theta =$$

$$\frac{\partial \forall}{\partial t} = \frac{\frac{1}{2} r \cdot r\, \delta\theta}{\delta t} = \frac{\frac{1}{2} r^2 (-\omega \delta t)}{\delta t} = -\frac{1}{2} r^2 \omega$$

$$\underline{V(r) = \tfrac{1}{2} r \omega\, \theta^{-1}}$$

Note:

$$\frac{\partial V(r)}{\partial t} = \tfrac{1}{2} r\omega (-\theta^{-2}) \frac{d\theta}{dt} = \tfrac{1}{2} r \omega^2 \theta^{-2}$$

Bernoulli (general)

$$\int_r^\ell \rho \frac{\partial V(r)}{\partial t} dr + \left[\frac{1}{2}\rho V(r)^2 + p(r) + \cancel{\rho g z}\right]_r^\ell = 0$$

or $$\int_r^\ell \rho \frac{\partial V(r)}{\partial t} dr + \frac{1}{2}\rho V_\ell^2 + p_\ell = \frac{1}{2}\rho V(r)^2 + p(r)$$

With $p_\ell = p_{atm} = 0$ and

$$V(r) = \frac{1}{2} r \omega \theta^{-1}, \quad V_\ell = \frac{1}{2} l \omega \theta^{-1}$$

$$p(r) = \frac{1}{8}\rho\left(\frac{\omega}{\theta}\right)^2(\ell^2 - r^2) + \int_r^\ell \rho \frac{1}{2} r(\omega/\theta)^2 dr =$$

$$\frac{1}{8}\rho\left(\frac{\omega}{\theta}\right)^2 \ell^2\left(1-\left(\frac{r}{\ell}\right)^2\right) + \frac{1}{4}\rho\left(\frac{\omega}{\theta}\right)^2 \ell^2\left(1-\left(\frac{r}{\ell}\right)^2\right)$$

Note: Unsteady effect on $p(r)$ is <u>TWICE</u> the value of steady result!

$$p(r) = \frac{1}{8}\rho\left(\frac{\omega \ell}{\theta}\right)^2\left(1-\left(\frac{r}{\ell}\right)^2\right) \cdot \begin{cases} 1 & \text{if steady only} \\ 3 & \text{if unsteady included.} \end{cases}$$

<u>Total Pressure Force</u> (per unit length into paper)

$$P_p = \int_0^\ell p(r) dr = \ell \int_0^1 \frac{1}{8}\rho\left(\frac{\omega \ell}{\theta}\right)^2\left(1-\left(\frac{r}{\ell}\right)^2\right) d\left(\frac{r}{\ell}\right) \cdot \begin{cases} 1 & St \\ 3 & US. \end{cases} =$$

$$\frac{1}{12}\rho\left(\frac{\omega \ell}{\theta}\right)^2 \ell \begin{cases} 1 & St. \\ 3 & USt. \end{cases}$$ <u>Directed upwards, perpendicular to plate</u> $\approx$ <u>Vertical since $\cos\theta \simeq 1$.</u>

Evaluation of Force on a Tire

$l = 7.8\,cm = 0.078\,m$; $\omega = 100\,rad/s$; $\rho = 1{,}000\,\frac{kg}{m^3}$

$$P_f = 395 \cdot \theta^{-2} \cdot \begin{cases} 1 & St. \\ 3 & USt. \end{cases} \quad N/m \quad \text{if } \theta \text{ in rad.}$$

With width of tire (into paper) $b = 20\,cm = 0.2\,m$ we have

$$\underline{P_{up} = P_f\, 0.2 = 79\,\theta^{-2}} \begin{cases} 1 & St. \\ 3 & USt. \end{cases} \; N.$$

Weight of a Car $\approx 1{,}000 \cdot 9.8 \approx 10^4\,N$

Force per wheel $\sim 2.5 \cdot 10^3\,N$ to overcome gravity

$$P_{up} = 79\,\theta_{cr}^{-2} \begin{Bmatrix} 1 & St. \\ 3 & USt. \end{Bmatrix} = 2.5 \cdot 10^3$$

gives

$$\theta_{cr}^2 = \frac{79}{2.5 \cdot 10^3} \begin{Bmatrix} 1 & St. \\ 3 & USt. \end{Bmatrix} \Rightarrow \underline{\theta_{cr} = \begin{cases} 0.177 & St. \\ 0.308 & USt. \end{cases}}$$

For any value of $\theta < \theta_{cr}$ the uplift force on a <u>smooth</u> tire hitting a <u>smooth</u> water-covered surface at a speed of $V_c = 65\,mph$ will exceed gravity, i.e. there is no contact force between tire and road. Car is hydroplaning! Now you know why tires have treads and why auto races stop whenever it rains.