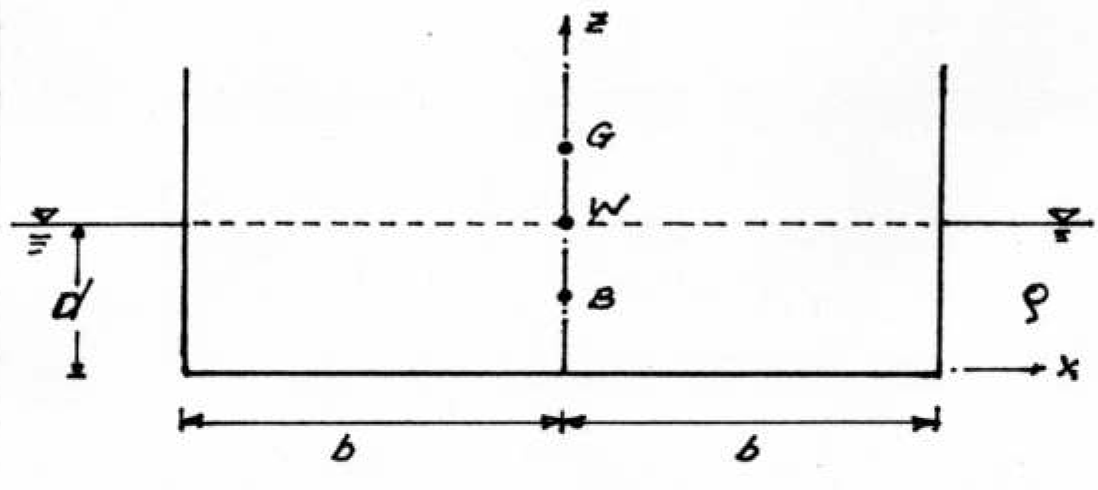
RECITATION #2

STABILITY OF A FLOATING BODY



The sketch above shows a 2-D body (ℓ into paper $\gg$ b and a) floating on the surface of a fluid of density ρ. The draft of the floating body is d and its width is $2b$. The body is assumed to be symmetrical around the vertical z-axis.

$$\left.\begin{aligned} &W_G = \text{weight of the floating body} \\ &G = \text{center of gravity of the floating body} \\ &\forall_B = \text{volume displaced by body} = 2bd \\ &B = \text{center of gravity of displaced fluid.} \end{aligned}\right\} (1)$$

We know from hydrostatics and force equilibrium that :

F_B = buoyancy force is vertically upwards ($\vec{F}_B = (0, F_{Bz})$), passes through B, and has a magnitude = weight of displaced fluid = $\rho g 2bd$ (per unit length into paper) (2)

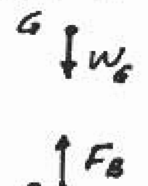
Also, for equilibrium,

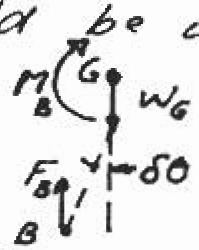
W_G = weight of floating body acts vertically downward ($\vec{W}_G = (0, -W_G)$), passes through G, and has a magnitude $= F_B = \rho g 2bd$ (3)

Both B and G are located on the vertical z-axis, i.e. at $x = 0$, because of symmetry, and we have the coordinates

$$(0, z_G) \text{ for } G, \quad (0, z_B) = \left(0, \frac{d}{2}\right) \text{ for } B \quad (4)$$

If $z_G > z_B$ and we turn the body a small angle around G (but retain the forces W_G and F_B) it would appear that the body would continue to turn, i.e. it would be unstable and roll over

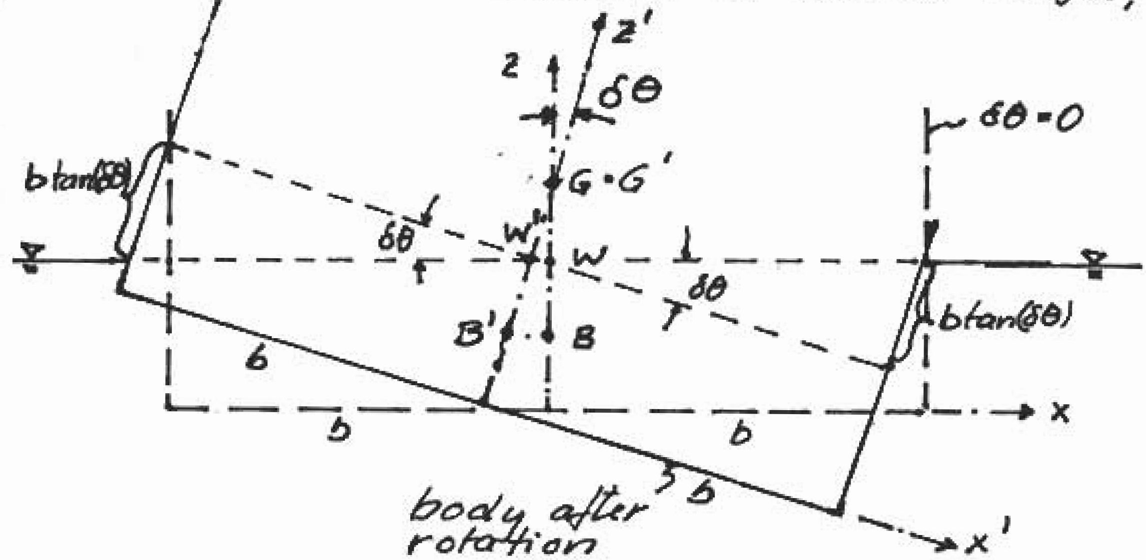




due to the moment of F_B $(= M_B = F_B(z_G - z_B)\delta\theta)$.

Is this really true?

To answer this question let us look at the forces acting on the floating body when this is rotated a small angle, $\delta\theta$,



around G, i.e. for the floating body's position shown in sketch above [note: the turning angle $\delta\theta$ is assumed small, but shown as "large" for clarity].

First we note that W, the location of the waterline on the original z-axis, has moved to a new location, W', after rotation. In the original (x, z)-coordinates simple geometry tells us that

$$\left.\begin{aligned} x_W - x_{W'} &= (z_G - z_W)\sin(\delta\theta) \\ \text{and} \quad z_W - z_{W'} &= (z_G - z_W)(\cos(\delta\theta) - 1) \end{aligned}\right\} \quad (5)$$

Thus, for a small angular rotation $\delta\theta \ll 1$

we have (Taylor Expansion! 18.01)

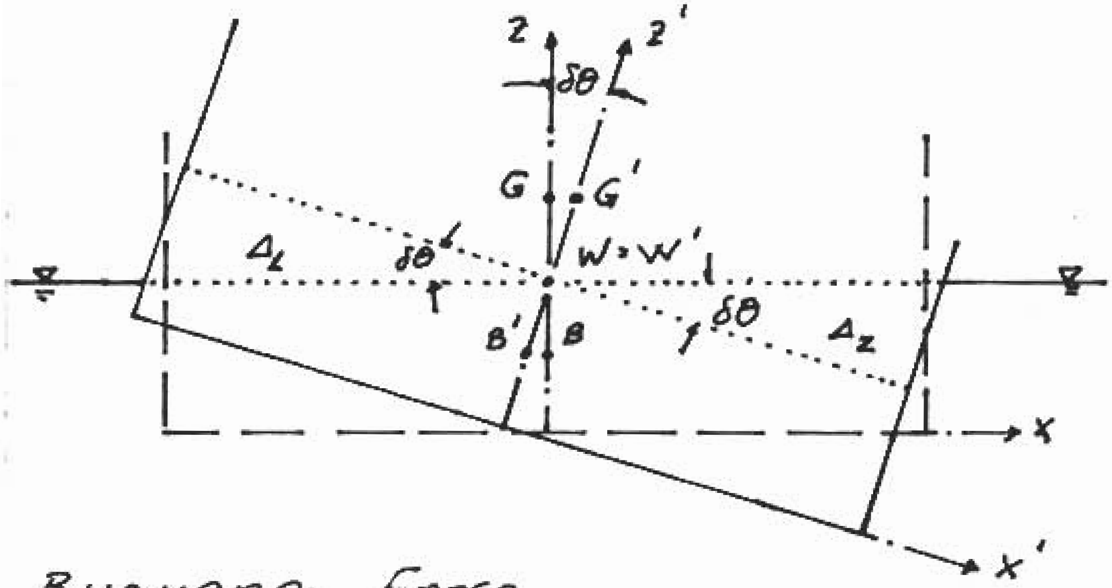
$$\left.\begin{aligned} \sin(\delta\theta) &= \delta\theta - \tfrac{1}{6}(\delta\theta)^3 + \cdots \approx \delta\theta \\ \cos(\delta\theta) &= 1 - \tfrac{1}{2}(\delta\theta)^2 + \cdots \approx 1 \end{aligned}\right\} \quad (6)$$

and (5) reduces to

$$\left.\begin{aligned} x_W - x_{W'} &\approx (z_G - z_W)\,\delta\theta \\ z_{W'} - z_W &\approx (z_G - z_W)\cdot 0 = 0 \end{aligned}\right\} \quad (7)$$

Therefore, the new position of the floating body may be obtained (<u>for small $\delta\theta$</u>) as a translation of the original position in the x-direction (by an amount of $x_W - x_{W'}$) followed by a rotation of the body around $W \equiv W'$ by an angle $\delta\theta$. Since only the (small) rotation affects the hydrostatic forces on the submerged part of the floating body, we may obtain these by simply rotating the floating body around the water-line point, W, and disregard the horizontal translation (which now shows up as a "translation" of the center of gravity G to G').

This statically equivalent position is shown in sketch "below".



Buoyancy Force

After rotation it is clear that a triangle Δ_1 no longer is submerged whereas the triangle Δ_2 has become submerged. Since

$$\text{Area } \Delta_1 = \text{Area } \Delta_2 = \frac{1}{2} b^2 \tan(\delta\theta) \qquad (8)$$

the submerged volume and hence the displaced volume is unchanged from its original value before rotation. Thus,

$$F_B = \rho g\, 2bd = W_G \qquad (9)$$

still holds, i.e. there is force equilibrium in the vertical direction. Note: Eq. (8) holds even if $\delta\theta$ were not $\ll 1$. However, if $\delta\theta$ were not small, the assumption that $z_W \simeq z_{W'}$, (7), would no longer hold and the body would not have $F_B = W_G$.

Center of Buoyancy

Taking the moment of the submerged portion of the rotated body around the z'-axis may again be treated in terms of the moment of the originally submerged (rectangular) volume and subtracting the moment of Δ_1 while adding the moment of Δ_2. Thus, with $\forall_B = 2bd$, cf (1), we have

$$(2bd)\,x'_{CB} = -(bd)\tfrac{1}{2}b + \Delta_1 \cdot \tfrac{2}{3}b$$

$$+(bd)\tfrac{1}{2}b + \Delta_2\tfrac{2}{3}b = \tfrac{2}{3}b^3\tan(\delta\theta)$$

or

$$x'_{CB} = \frac{1}{3}\frac{b}{d}\,b\tan(\delta\theta) \qquad (10)$$

with x'_{CB} denoting the x'-value for the center of buoyancy in the rotated state.

Similarly, moment around the x'-axis gives

$$(2bd)\,z'_{CB} = (2bd)\tfrac{1}{2}d - \Delta_1\left(d - \tfrac{1}{3}b\tan(\delta\theta)\right)$$

$$+\Delta_2\left(d + \tfrac{1}{3}b\tan(\delta\theta)\right) = 2bd\,\tfrac{1}{2}d + \Delta_1\tfrac{2}{3}b\tan(\delta\theta)$$

or

$$z'_{CB} = \frac{1}{2} d + \frac{1}{6} \frac{b}{d} b \tan^2(\delta\theta) \qquad (11)$$

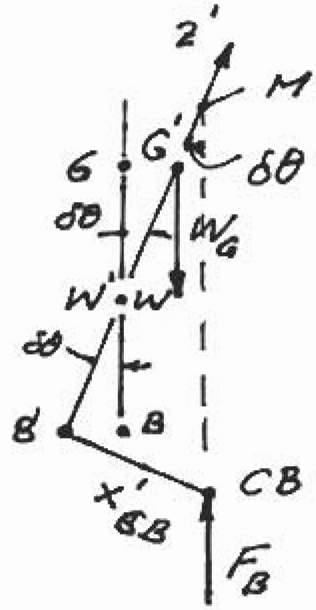
Recalling that we previously made use of the small angle assumption, $\delta\theta \ll 1$, in order to neglect any vertical movement of W, cf (5) and (6). This amounts to a neglect of terms that are of order $(\delta\theta)^2$, and since $\tan(\delta\theta) = \sin(\delta\theta)/\cos(\delta\theta) \approx \delta\theta/1 + O(\delta\theta^3) \approx \delta\theta$, Eq. (11) reduces to

$$z'_{CB} = \frac{1}{2} d \qquad (12)$$

since we, for consistency, should disregard the second term in (11) as being of order $\tan^2(\delta\theta) = O(\delta\theta)^2$ and therefore negligibly small.

Stability Condition

Since the buoyancy force F_B acts vertically upwards and passes through the center of buoyancy, CB, whereas the weight of the body, $W_G = F_B$, acts vertically downwards and passes through G, the forces acting on the rotated floating body is simply a moment acting around the y axis that points into the paper)



Aided by the sketch above, simple geometry yields, for the point M = the <u>METACENTER</u>,

$$z'_M = z'_{B'} + x'_{CB}/\tan(\delta\theta) =$$

$$z'_{B'} + \frac{1}{3}\frac{b}{d}b$$

or

$$z'_M - z'_B = z'_M - z_B = \frac{1}{3}\frac{b}{d}b \qquad (13)$$

We can now state the conditions for stability of a floating body as follows:

In words:

If the METACENTRIC HEIGHT is ABOVE the CENTER OF GRAVITY of the floating body, the body is stable for small rotation angles, since buoyancy acts to restore original position.

In equation form:

If

$$z_G - z_B < \frac{1}{3}\frac{b}{d}\,b^{*} \qquad (14)$$

a floating body is stable for small rotation angles.

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* This expression was for a particular shape [a box-shape]. The general expression, for a symmetric body, is

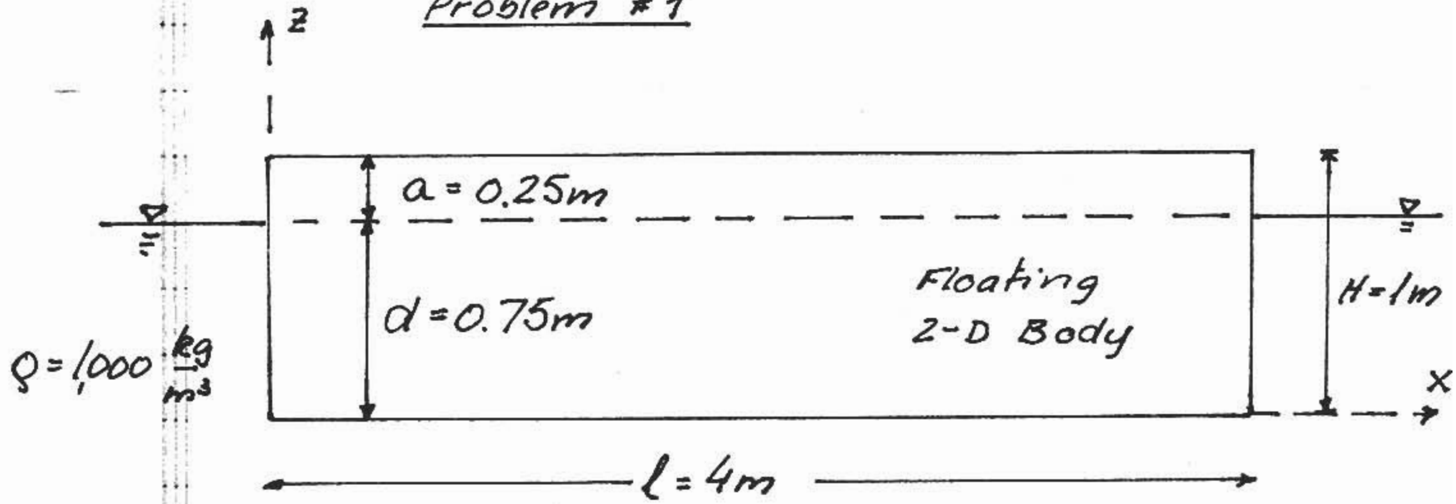
$$z_G - z_B < \frac{I_{yy}}{\forall_B} \qquad (15)$$

where

$$\left.\begin{aligned} &\forall_B = \text{displaced volume of fluid} \\ &I_{yy} = \int_{A_w} x^2\, dA = \text{Inertia Moment around } y \\ \text{of}\quad &A_w = \text{waterline area of floating body} \end{aligned}\right\} (16)$$

RECITATION # 2

Problem # 1



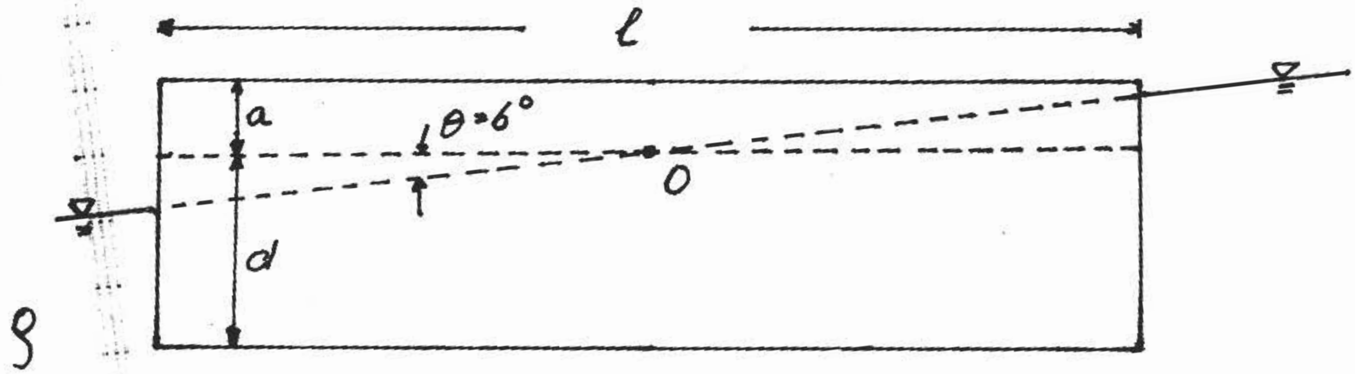
1) Determine the density of material of floating body, ρ_B. What do you think it is?

2) Determine Center of gravity of floating body, (X_G, Z_G)

3) Determine center of gravity of displaced volume of water, (X_B, Z_B)

Problem #2

Same 2-D Body as in Problem #1, but rotated $\theta = 6°$ around centerline of waterline, O.
$(\cos\theta = 0.995 \approx 1.0$; $\sin\theta \approx \tan\theta \approx 0.1)$



1) Determine the net horizontal force on the rotated floating body.

2) Determine the uplift (buoyancy) force on the rotated floating body.

3) Is the rotated floating body in equilibrium?

RECITATION #2

Problem #1

a)

Body is in equilibrium. Weight of body = $W_B = \rho_B g \cdot (a+d)\ell$, = Buoyancy force = weight of displaced fluid = $F_B = \rho g d \cdot \ell$.

$$\underline{\rho_B} = \rho \frac{d}{a+d} = \underline{0.75 \rho} = \underline{750 \frac{kg}{m^3}} \quad \underline{(\sim Wood!)}$$

b)

Rectangle CG is right ~~sm~~smack in the middle, i.e.

$$\underline{X_G} = \frac{\ell}{2} = \underline{2m}\ ; \quad \underline{Z_G} = \frac{a+d}{2} = \frac{1}{2} = \underline{0.5m}$$

c)

Same thing, except only submerged portion of floating body is considered.

$$\underline{X_B} = \frac{\ell}{2} = \underline{2m}\ ; \quad \underline{Z_G} = \frac{d}{2} = \frac{0.75}{2} = \underline{0.375m}$$

Notice: Net horizontal force is zero. On left part of body $\frac{1}{2}\rho g d^2$ pushes towards the right; and is balanced by $\frac{1}{2}\rho g d^2$ acting towards the left from the right part of body. Notice also: Lines of action of vertical forces W_B & F_B coincide ($X_G = X_B$), and so do the lines of action of horizontal forces ($z_{Hor.For} = \frac{1}{3}d$)

Problem No: 2

a)

Horizontal force (to the right) = hydrostatic force (to the right) on body's projection onto a vertical plane. This vertical plane extends from the free surface down to the deepest corner, which is a distance of

$$d_{max} = \left(d + \frac{\ell}{2}\cdot\tan\theta\right)\cos\theta \approx d + \frac{\ell}{2}\theta = (0.75+0.2)m$$

Same way for force to the left & same result.

<u>Net Horizontal force = 0</u>

b)

Compared to original $\theta = 0$ position an extra triangular volume $\left(\frac{1}{2}\cdot(\ell/2)\cdot(\ell/2)\tan\theta\right.$ has been submerged (dashed). But an exact replica (dashed) is no longer submerged. So the volume displaced by the rotated body is identical to that of the body when $\theta = 0$. Same buoyancy force as before, i.e. same as weight of body

<u>$F_B = W_B$ (Vertical Force Equilibrium!)</u>

c)

Horizontal forces balance! Vertical forces balance! Looks as if body is in equilibrium. But wait - there is more to equilibrium than forces! How about MOMENT balance?

This is the topic referred to a

<u>STABILITY OF FLOATING BODIES</u>