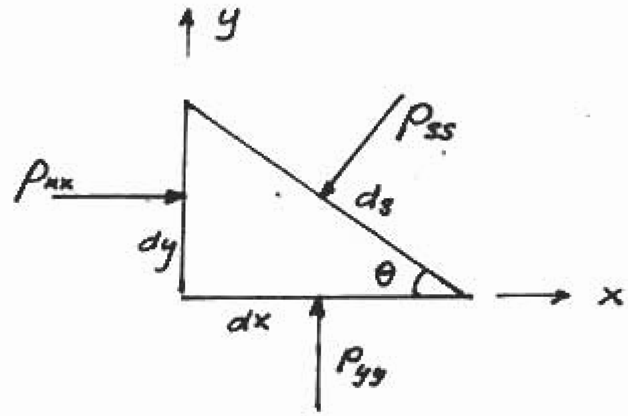
RECITATION #1

REVIEW OF HYDROSTATICS

A fluid at rest can, by definition, not support any shear stress.



Pressure is positive for compression – arrows point inwards from the outside if $p > 0$ – and acts $\perp$ to boundary

x-direction force equilibrium:

$$p_{xx}\,dy - (p_{ss}\,ds)\sin\theta = p_{xx}(\sin\theta \cdot ds) - p_{ss}(\sin\theta \cdot ds) = 0$$

$$\underline{p_{ss} = p_{xx}}$$

y-direction force equilibrium:

$$p_{yy}\,dx - (p_{ss}\,ds)\cos\theta = p_{yy}(\cos\theta\, ds) - p_{ss}(\cos\theta\, ds) = 0$$

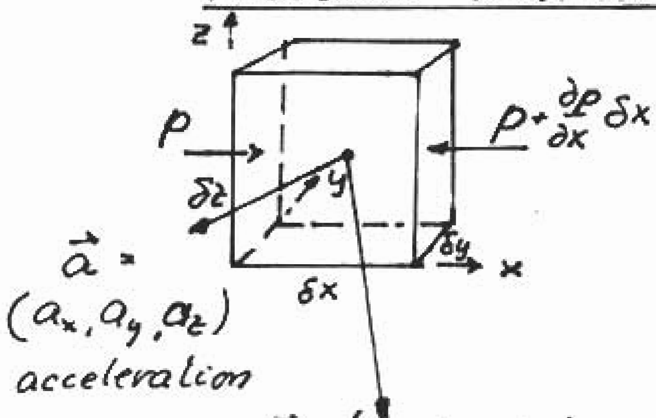
$$\underline{p_{ss} = p_{yy} = p_{xx}} \quad \underline{(\text{independent of } \theta)}$$

- Pressure at a point is isotropic – same in all directions

- Pressure is positive for compression

- Pressure is alway normal to surface upon which it acts.

<u>Pressure Variation</u>



$\delta \forall = \delta x \delta y \delta z$ = volume of particle

$\rho \delta \forall$ = mass of particle

$\rho \vec{g} \delta \forall$ = gravity force on particle

$\vec{g} = (g_x, g_y, g_z)$ = gravity

$\rho \vec{a} \delta \forall$ = mass × acceleration of particle

$$[\,p - (p + (\partial p/\partial x)\delta x\,)]\,\delta y \delta z = -\frac{\partial p}{\partial x}\delta \forall =$$

net pressure force on particle in x-direction

<u>Newton's Law</u> (in x-direction)

$$\rho\, a_x \cancel{\delta \forall} = -\frac{\partial p}{\partial x}\cancel{\delta \forall} + \rho g_x \cancel{\delta \forall} \qquad \text{or}$$

$$\frac{\partial p}{\partial x} = \rho(g_x - a_x)$$

and analogous expression in y- and z-directions

$$\frac{\partial p}{\partial y} = \rho(g_y - a_y) \;;\; \frac{\partial p}{\partial z} = \rho(g_z - a_z)$$

In vector form:

$$\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) = \text{grad}\, p = \nabla p = \rho(\vec{g} - \vec{a}) \qquad (1)$$

Note: We did not account for any shear stresses in the fluid in evaluating the forces on δV. Thus, the fluid is either <u>inviscid</u> ($\mu = \nu = 0$) and the equations above are known as the Euler-equations or its motion is that of a solid body, e.g. experiencing a uniform acceleration or an angular rotation.

<u>Hydrostatics ($\bar{a} = 0$)</u>

In the absence of any acceleration and subject only to gravitational forces, we choose a convenient coordinate system with

z - positive upwards opposite of g = gravitational acceleration = $9.8\ m/s^2$.
(x, y) - horizontal plane

With this choice, we have

$$\vec{g} = (0, 0, -g) \qquad (2)$$

and (1) becomes

$$\left.\begin{aligned} \frac{\partial p}{\partial x} = 0 \;\Rightarrow\; p = p(y, z) \\ \frac{\partial p}{\partial y} = 0 \;\Rightarrow\; p = p(x, z) \end{aligned}\right\} p = p(z)$$

and

$$\frac{\partial p}{\partial z} = -\rho g \;\Rightarrow\; \underline{p = -\int_{z_0}^{z} \rho g \, dz + p_0}$$

where

$$\underline{p_0 = \text{pressure at } z = z_0}$$

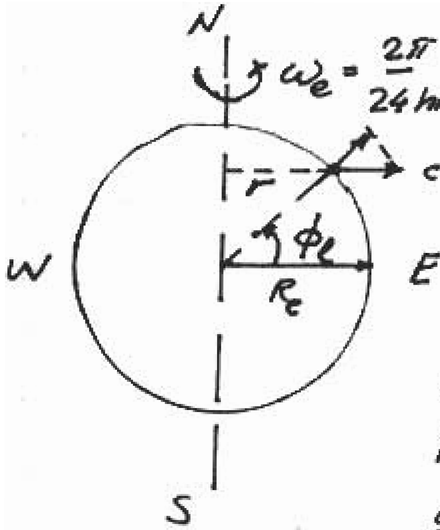
With the same fluid $\rho \cong$ constant if the fluid is incompressible (excellent assumption in most cases - see Lecture #2)

Earth's gravity results from gravitational attaction

$$g_g \propto r^{-2}$$

where r is the distance from the center of the Earth. With Earth's radius, $R_e \sim 6{,}360$ km a change in elevation, say $r \pm 10$ km, would have a negligible effect on the value of g_g ($\pm 0.3\%$)

However, Earth's surface is not an inertial coordinate system since Earth is rotating around its axis. Thus, a point on Earth's surface moves in a circular path of radius $r \sim R_e \cos\phi_e$ (ϕ_e = lattitude).



The combination of gravitational and centrifugal accelerations gives us the effective gravity

$$g_e = g \cong g_g - \omega_e^2 r \cos\phi_e = g_g - \omega_e^2 R_e \cos^2\phi_e$$

Thus, g is larger at the poles ($\phi_e = 90°$) than at the Equator ($\phi_e = 0°$) by $\sim 0.034\, m/s^2$ or $\sim 0.3\%$ of the "standard" value of $g = 9.8\frac{m}{s^2}$ (9.806 or $9.81\, m/s^2$, but 9.8 is close enough for our purposes).

Thus, for an incompressible fluid we have

$\underline{\rho \simeq \text{constant}}$; $\underline{g \simeq \text{constant}}$

and therefore

$$p = -\int_{z_0}^{z} \rho g \, dz + p_0$$

or

$$\underline{p + \rho g z = p_0 + \rho g z_0 = \text{CONSTANT}}$$

which is known as <u>HYDROSTATIC PRESSURE DISTRIBUTION</u>

REVIEW OF DIMENSIONAL ANALYSIS

An equation describing a physical system's behavior must be <u>dimensionally homogeneous</u>, i.e. a statement like

$$a = b + c$$

makes sense only if the dimensions of a, b, and c are identical. Denoting the dimensions of a quantity by [ ] we require that

$$[a] = [b] = [c]$$

In Newtonian Mechanics all "units" may be expressed in terms of three (3) fundamental dimensions:

L = length unit
M = mass unit
T = time unit

For example, we have from Newtons Law

Force = Mass $\cdot$ Acceleration

and therefore

$$[\text{Force}] = [\text{Mass}] \cdot [\text{Acceleration}] =$$

$$M \frac{L}{T^2} = M L T^{-2}$$

So, the units of a force

$$[F] = M L T^{-2}$$

and the units of a stress (= force per unit area)

$$[\text{Stress}] = \left[\frac{F}{A}\right] = \left[\frac{F}{L^2}\right] = M L^{-1} T^{-2}$$

In a standardized system of basic units, e.g. SI-system, we may give special names to units that are derived from the fundamental units (m, kg, s for SI-system)

$$[F] = M L T^{-2} = kg \cdot m / s^2 = 1\, N(\text{ewton})$$

$$[\text{Stress}] = M L^{-1} T^{-2} = kg/(m \cdot s^2) = 1 \frac{N}{m^2} = 1\, Pa(\text{scal}).$$

So long as we use a single system of basic units we need not worry about the dimensions of different quantities in an expression, i.e. we can treat them as "numbers", since the units will take care of themselves – provided the expression to be evaluated is dimensionally correct.

Recommendation

- When formulating a problem use symbolic notation, e.g. take density as "ρ" and gravity as "g" (NOT 1,000 (kg/m³) or 9.8 (m/s²)).

- Carry out all necessary manipulations and operations retaining symbolic notations until the answer is obtained (in symbolic notation)

- Perform a check, by introducing the dimensions of quantities appearing in your answer, to show that your answer is dimensionally correct.

- Give the values of all quantities in your answer in a consistent set of units (e.g. SI-system)

- Then, obtain the numerical answer to the problem, and include the units of your answer, e.g. $F_x = 3.2$ N.

Problem

As an example, consider the problem of determining the pressure in water at a distance of 3 ft below a free surface.

Answer:

Denoting pressure by "p", density of fluid by "ρ", acceleration due to gravity by "g", and distance below free surface (where "p" = 0) by "h" we have from hydrostatics

$$p = \rho g h$$

Check of dimensions:

$$[p] \stackrel{?}{=} [\rho][g][h] = \frac{M}{L^3} \cdot \frac{L}{T^2} \cdot L = M L^{-1} T^{-2}$$

Answer is dimensionally correct since p is a stress $\rightarrow [p] = M L^{-1} T^{-2}$!

Numerical values in consistent units (SI)

$\rho = 1{,}000\ kg/m^3$; $g = 9.8\ m/s^2$;
$h = 3$ feet $= 0.91m$ (to get it in SI-units)

Numerical Answer:

$$p = \rho g h = 1{,}000 \cdot 9.8 \cdot 0.91 = \underline{8{,}920 Pa}$$

Notice: we treat ρ, g, h as "numbers" and we know that the "number" we get for "p" is in the units of a stress, i.e. Pa(scal) in SI!

Dimensional Analysis

To illustrate the concept and prowess of dimensional analysis we take a concrete example:

The experimental determination of the drag force, F_D, for a fixed smooth-surfaced sphere of diameter D due to a flowing fluid of density ρ and kinematic viscosity ν (dynamic viscosity $\mu = \rho\nu$) moving a velocity V towards the sphere.

When the "problem" is cast in terms of an "experimental" problem, it is easier to visualize the nature of the variables we introduced above

1) The dependent variable is what we are looking for. This is F_D in the example.

2) The independent variables are all the physical quantities we expect to play a role in determining to value of the dependent variable, i.e. things we can change in our experimental set-up. In the example, we can change the size of the sphere, D, the flowing fluid (identified by ρ and ν), and its approach velocity, V.

If we have correctly identified all the independent variables that may affect the outcome of our experiment (the drag force, F_D), then simple physical reasoning suggests that

$$\underline{F_D} = \text{function of } D, \rho, \nu, \text{ and } V = \underline{F_D(D, \rho, \nu, V)}$$

At first sight, this appears to be a daunting problem, since it suggests that we need to perform experiments in which fluid (ρ & ν) and velocity (V) are kept constant and the sphere's diameter is changed from one experiment to the next. Then another set of experiments, with a different fluid, then another set with different velocities, etc. One could spend a lifetime on this problem, and still leave plenty of work left to be done by generations to come.

However, use of <u>DIMENSIONAL ANALYSIS simplifies our problem considerably</u>.

The basic notion behind dimensional analysis is that <u>THE PROBLEM</u> (not some government institution or international convention) <u>DETERMINES THE BASIC FUNDAMENTAL DIMENSIONS</u>.

$$\nu = \pi_1 \, DV \Rightarrow \pi_1 = \text{a number} = \frac{\nu}{DV}$$

$$F_D = \pi_D \, \rho D^2 V^2 \Rightarrow \pi_D = \text{a number} = \frac{F_D}{\rho D^2 V^2}$$

and therefore, with π_1 = dimensionless kinematic viscosity and π_D = dimensionless drag force, we have

$$\pi_D = \pi_D(1, 1, 1, \pi_1) = \pi_D(\pi_1)$$

Thus, the use of problem-specified basic units, the experimental problem we are facing has been reduced from

Drag Force = Function of Four (4) Variables

to

Non-dimensional Drag Force $= \pi_D = \dfrac{F_D}{\rho D^2 V^2} =$

Function of One (1) variable $= \pi_D\left(\pi_1 = \dfrac{\nu}{DV}\right)$

or

$$\underline{F_D} = \underline{\pi_D\left(\pi_1 = \frac{\nu}{DV}\right) \rho D^2 V^2}$$

All we need to determine experimentally is the dependency of π_D on a single variable $\pi_1 = \nu/(DV) = Re_D^{-1}$ (Re_D = Reynolds Number) and our "problem" is solved!!