Summary of Finite Control Volume Analysis in Fluid Mechanics

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At this point we have developed a set of tools from basic conservation principles for use in the analysis of fluid mechanics problems.

We have developed (derived) these tools (equations) by applying fundamental conservation laws (e.g. conservation of mass, volume, linear momentum and energy) to finite control volumes as opposed to elementary representative volumes (ERVs). When properly applied, our finite control volume tools will result in the determination of bulk features of the fluid flow - the discharge, Q, the average velocity, $V = Q/A$, and the total integrated force due to pressures and shear stresses, F_p and F_τ, etc. These bulk features are often sufficient to solve practical problems in *hydraulics*. If detailed flow features such as velocity distributions, $v(y)$, pressure distributions, $p(x, y)$, and shear stress distributions, $\tau(x, y)$, are required, then ERV analysis based on differential equations must be employed (i.e. the tools of *hydrodynamics*).

The purpose of this write-up is to present a concise summary of the tools we have at our disposal for the analysis of fluid flow problems, along with some simplifications and guidance that should make it easier to apply these tools in a proper fashion.

1 Picking the Control Volume

In choosing the control volume for analysis, the most important decision is to pick inflow and outflow areas, A_{in} and A_{out}, where the flow is well behaved.

Well behaved flow implies that streamlines are *straight* and *parallel*. The flow area, A, is chosen so that it is perpendicular to the streamlines. With this choice, the pressure varies hydrostatically in the direction perpendicular to the streamlines. Thus, the pressure varies linearly over the flow area A, and the pressure force on A is *perpendicular to* A with a magnitude of

$$P = \int_A p\, dA = p_{\text{CG},A} \cdot A \tag{1.1}$$

where

$$p_{\text{CG}} = \text{ pressure at center of gravity (CG) of } A. \tag{1.2}$$

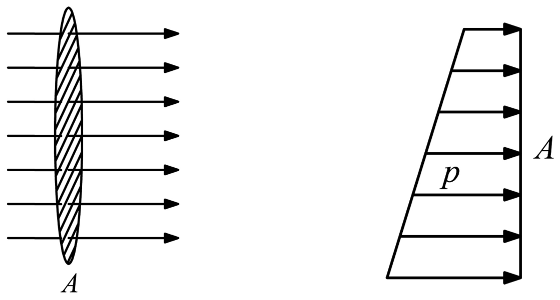
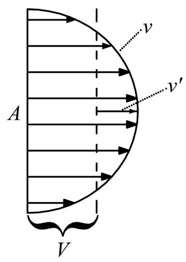


Figure 1: Straight and parallel streamlines passing through the flow area (left); linearly varying pressure (right).



Also, if the flow is well behaved, it must be nearly uniform in the vicinity of A such that the detailed velocity variation across A, v, will not be excessive. Thus, we have

$$\begin{aligned} Q &= \text{volume flow rate across } A = \text{Discharge} \\ &= \int_A v\, dA = VA \end{aligned} \tag{1.3}$$

which defines the cross-sectional average velocity

$$V = Q/A = \text{average velocity.} \tag{1.4}$$

One can, with reasonable certainty, expect the local deviation of the velocity, $v' = v - V$, to be relatively small compared to V over most of A. For example, if $|v'/V| = \delta_m \approx 0.1$ over most of A, the momentum coefficient

$$K_m = 1 + \frac{1}{A}\int_A \left(\frac{v'}{V}\right)^2 dA \approx 1 + \delta_m^2 \approx 1 + 0.01 \approx 1,$$

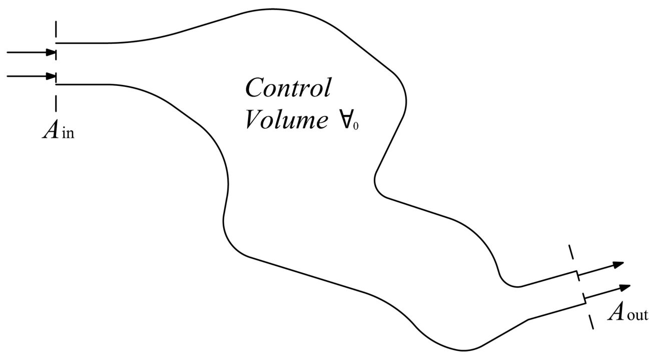
and the energy coefficient

$$K_e \approx 1 + 3\delta_m^2 \approx 1 + 0.03 \approx 1$$

can, without actually knowing v, safely be taken as

$$K_m = K_e \approx 1. \tag{1.5}$$

These are the features that make the choice of inflow and outflow areas where flow is well-behaved a *must* when picking your finite control volume. The control volume itself is everything between the inflow and outflow areas occupied by the fluid of interest.



Notes of caution:

1. If inflow and/or outflow areas have a free surface, the pressure starts at $p = 0$ at the free surface, and $p_{\text{CG}} = \rho g h_{\text{CG}}$ where h_{CG} is the vertical distance of CG of A below the free surface. Thus, p_{CG} is known if the location of free surface is known.

2. If the flow is in a closed conduit, the pressure at the top of the conduit is p_0 and this is not (necessarily) zero. So for a closed conduit (pipe) flow, $p_{\text{CG}} = p_0 + \rho g h_{\text{CG}}$, where h_{CG} is the vertical distance from the top of the conduit to CG of A. This is not known unless p_0 is known!

3. If the flow area is a freely falling jet such that A has air around it and nothing to support its weight (such as a "bottom"), then the effect of gravity makes the fluid accelerate in the vertical direction, $a_z = -g$, and the pressure is $p_{\text{atm}} = 0$ throughout the fluid. Thus $p_{\text{CG}} = 0$ as long as A is chosen where flow is well behaved [at vena contracta].

2 Conservation of Mass and Volume

With $\dot{M}$ = rate of mass flow across $A = \int_A \rho v \, dA$, mass conservation for a finite control volume states:

$$\begin{aligned} \sum \dot{M}_{\text{in}} - \sum \dot{M}_{\text{out}} &= \frac{\partial}{\partial t} [\text{mass of fluid in CV}] \\ &= 0 \text{ (for steady conditions).} \end{aligned} \tag{2.1}$$

If ρ is constant and the fluid is incompressible, conservation of volume (a.k.a. *continuity*) states:

$$\begin{aligned} \sum Q_{\text{in}} - \sum Q_{\text{out}} &= \frac{\partial}{\partial t} [\text{volume of fluid in CV}] \\ &= 0 \text{ (for steady conditions).} \end{aligned} \tag{2.2}$$

2.1 Continuity equation for a stream tube/conduit

The following applies only for one inflow area and one outflow area, and only under steady conditions:

$$Q_{\text{in}} = Q_{\text{out}} = Q = \text{Discharge} = \text{Constant} \tag{2.3}$$

and with Q given by (1.3),

$$Q = VA \quad \Rightarrow \quad \text{Constant} \quad \Rightarrow \quad V = \frac{Q}{A}. \tag{2.4}$$

Thus, along a streamline tube (conduit) of varying cross-sectional area, A, the velocity V varies inversely proportional to A. This allows us to relate velocities at two flow areas

$$Q = V_1 A_1 = V_2 A_2 \quad \Rightarrow \quad V_1 = \frac{A_2}{A_1} V_2 \quad \text{or} \quad V_2 = \frac{A_1}{A_2} V_1. \tag{2.5}$$

If Q is known we can use (2.4) to obtain V wherever A is known. If Q is *not* known, we can use (2.5) to express V_1 in terms of V_2 (or vice versa) thereby reducing the number of unknowns by one.

Note: Continuity, i.e. $Q = VA$, is virtually always one of the tools that you need to employ when solving a fluid mechanics problem.

3 Conservation of Momentum

Conservation of momentum for a finite control volume under steady conditions states that

$$\sum \overrightarrow{\text{MP}} + \left\{ \begin{array}{l} \overrightarrow{\text{gravity force}} \\ \text{on fluid in CV} \end{array} \right\} + \left\{ \begin{array}{l} \text{all other } \overrightarrow{\text{forces}}, \\ \text{on fluid in CV} \end{array} \right\} = \frac{\partial}{\partial t} \left\{ \int_{\text{CV}} \rho \vec{q} d\forall \right\} = 0 \tag{3.1}$$

where

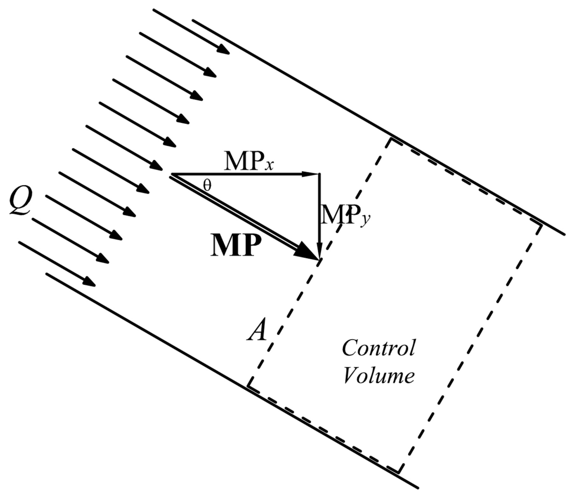
$$\text{MP} = \text{thrust at flow area} = \left(K_m \rho V^2 + p_{\text{CG}} \right) A \tag{3.2}$$

is perpendicular to A and points inwards (i.e. *towards* the CV), regardless of whether the flow area is an inflow or outflow area. Note that $K_m \approx 1$ (see 1.5).

Since the momentum principle (3.1) deals with forces, it is obviously of use when looking for estimates of forces from a fluid on its surroundings

$$\sum \left\{ \begin{array}{l} \text{all other } \overrightarrow{forces} \text{ on fluid} \\ \text{in CV } \textit{from} \text{ surroundings} \end{array} \right\} = - \sum \left\{ \begin{array}{l} \text{all other } \overrightarrow{forces} \text{ on fluid} \\ \text{in CV } \textit{on} \text{ surroundings} \end{array} \right\} \tag{3.3}$$

or, it may be used if "all other $\overrightarrow{\text{forces}}$" and the $\overrightarrow{\text{gravity force}}$ are known (or can be estimated) to obtain a relationship between the MPs, i.e. velocities and pressures, at inflow/outflow areas.



Important: (3.1) is a *vector equation* so it has 3 components (one in each of x, y, and z).

$$\text{MP}_x = \text{MP}\cos\theta \tag{3.4}$$
$$\text{MP}_y = -\ \text{MP}\ \sin\theta$$

When can gravity be neglected?

1. When forces in the (x, y)-plane are desired since $\vec{g}$ acts in the negative z-direction.
2. When the gravity force $= \rho g \forall_0 \ll$ momentum force $= \rho V^2 A$, i.e. when

$$\frac{\forall_0/2}{A} = \text{a length} \ll \frac{V^2}{2g} = \text{velocity head.} \tag{3.5}$$

When can shear stress (frictional) forces be neglected?

1. When the frictional force is $\ll$ momentum force, i.e. when

$$\tau_s A_\tau \ll \rho V^2 A$$

or with

$$\tau_s = \frac{1}{8}\rho f V^2$$

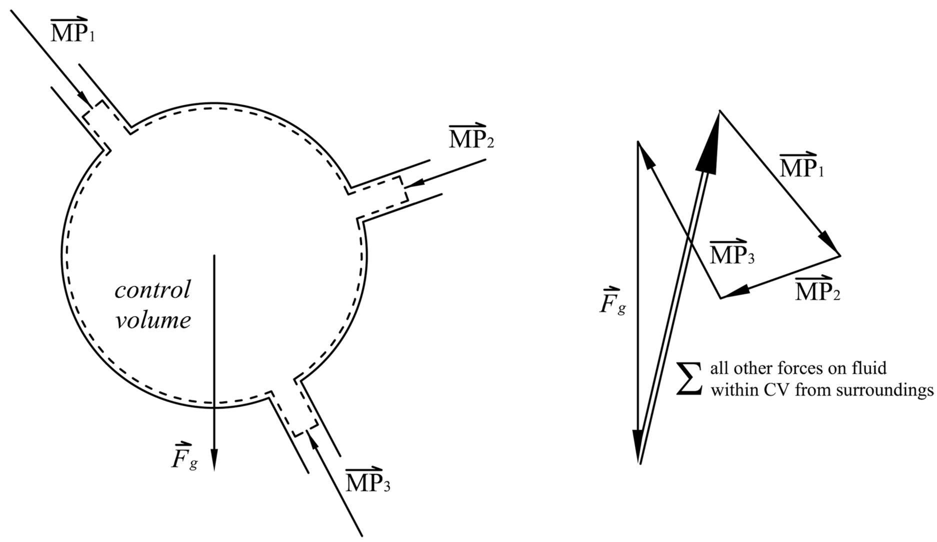
when

$$\frac{A = \text{flow area}}{A_\tau = \text{wall area}} \gg \frac{f}{8} \approx \frac{0.02}{8} = \frac{1}{400}. \tag{3.6}$$

where f is the Darcy-Weisbach friction factor ≈ 0.02. This corresponds to a "short" transition.

The momentum principle is used:

1. to compute forces when MPs are known,
2. to solve for ps and/or Vs when forces are "known" (also need continuity),
3. whenever there is an unknown head loss (expanding flow) and forces may be approximately evaluated (or neglected).



4 Energy Conservation - Bernoulli Principle

$$\begin{aligned}\dot{E} &= \text{rate of flow of mechanical energy across } A \\ &= \rho g Q H\end{aligned} \tag{4.1}$$

$$H = \text{Total Head} = K_e \frac{V^2}{2g} + \frac{p_{\text{CG}}}{\rho g} + z_{\text{CG}} \tag{4.2}$$
where $K_e \approx 1$ [see (1.5)]

$$\left.\begin{aligned}&\frac{V^2}{2g} = \frac{(Q/A)^2}{2g} = \text{Velocity Head} \\ &\frac{p_{\text{CG}}}{\rho g} = \text{Pressure Head } (p_{\text{CG}} = \text{pressure at CG of } A) \\ &z_{\text{CG}} = \text{Elevation Head } (z_{\text{CG}} = \text{elevation of CG of } A)\end{aligned}\right\} \tag{4.3}$$

4.1 Conservation of Mechanical Energy

$$\sum \dot{E}_{\text{in}} - \sum \dot{E}_{\text{out}} = \dot{E}_{\text{diss}} + \frac{\partial}{\partial t}\int_{\text{CV}} \left(\rho g z + \frac{1}{2}\rho \vec{q}^{\,2}\right) d\forall \tag{4.4}$$
$$= \dot{E}_{\text{diss}} (\text{for steady conditions})$$

$$\frac{\partial}{\partial t}\int_{\text{CV}} \left(\rho g z + \frac{1}{2}\rho \vec{q}^{\,2}\right) d\forall = \text{rate of change of mechanical energy within CV}$$

$$\dot{E}_{\text{diss}} = \text{rate of dissipation of mechanical energy within CV} \tag{4.5}$$

For a streamtube (conduit) having one inflow area and one outflow area:

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = \rho g Q_{\text{in}} H_{\text{in}} - \rho g Q_{\text{out}} H_{\text{out}} = \dot{E}_{\text{diss}} \tag{4.6}$$

but $Q_{\text{in}} = Q_{\text{out}} = Q$ (2.3), so

$$H_{\text{in}} - H_{\text{out}} = \frac{\dot{E}_{\text{diss}}}{\rho g Q} = \Delta H_{\text{CV}} \tag{4.7}$$

$$\Delta H_{CV} = \text{head loss within CV} = \frac{\dot{E}_{diss}}{\rho g Q} \tag{4.8}$$

4.2 Generalized Bernoulli Principle

Flow is from (1) to (2):

$$H_1 = H_2 + \sum_{1 \to 2} \Delta H \tag{4.9}$$

4.2.1 Frictional Head Loss (in straight sections of length ℓ)

$$\Delta H_f = \frac{\tau_s P}{\rho g A}\ell = f \frac{\ell}{(4R_h)} \frac{V^2}{2g} \tag{4.10}$$

$$f = f\left(\text{Re} = \frac{V(4R_h)}{\nu}, \frac{\varepsilon}{(4R_h)}\right) \tag{4.11}$$
$$= \text{Darcy-Weisbach Friction Factor (from Moody Diagram)}$$

$$R_h = \text{Hydraulic Radius} = \frac{\text{Flow Area}}{\text{Wetted Perimeter}} = \frac{A}{P} \tag{4.12}$$
$$4R_h = D = \text{ pipe diameter if the conduit is a circular pipe}$$

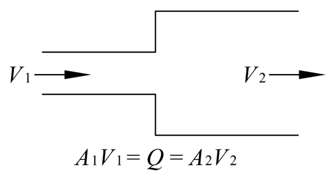
4.2.2 Minor Head Loss

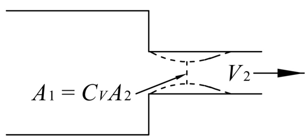
$$\Delta H_m = K_L \frac{V^2}{2g} \tag{4.13}$$

$$K_L = \text{minor loss coefficient} \tag{4.14}$$

Minor losses are associated with expansions of flow.

$$\left.\begin{aligned} \Delta H_{exp} = \frac{(V_1 - V_2)^2}{2g} = \left(1 - \frac{A_1}{A_2}\right)^2 \frac{{V_1}^2}{2g} = \left(\frac{A_2}{A_1} - 1\right)^2 \frac{{V_2}^2}{2g} \\ \text{If } A_2 \gg A_1 \text{ (outflow to large reservoir): } \Delta H_m = \frac{{V_1}^2}{2g} \\ A_1/A_2 = C_V = \text{contraction coefficient for inflow with sharp edges:} \\ \Delta H_m = \left(\frac{1}{C_V} - 1\right)^2 \frac{{V_2}^2}{2g} \end{aligned}\right\} \tag{4.15}$$





- *When is $H_1 \approx H_2$?* or *When is $\Delta H_{1\to 2} \approx 0$?*
  $\Delta H \approx 0$ for a short transition of a converging flow. Short means that

  $$\Delta H_f = f \frac{\ell}{4R_h} \frac{V^2}{2g} \ll \frac{V^2}{2g} \text{(friction is negligible).}$$

  $$\frac{\ell P}{A} = \frac{\text{wall friction area}}{\text{flow area}} \ll \frac{4}{f} \cong 200 \tag{4.16}$$

  [Note: $\sim$ same as for momentum, (3.6)] Converging flow means velocity increases (pressure decreases) in the direction of flow.

- *When is $\Delta H_{1\to 2} \neq 0$?*
  Whenever the distance from (1) to (2) is not short (c.f. (4.16)) and frictional losses therefore are non-negligible; or whenever there is a flow expansion (decrease in velocity in flow direction) between (1) and (2).

- *When can $\frac{V^2}{2g}$ be neglected?*

$$V_1 = \frac{A_2}{A_1} V_2 \quad \Rightarrow \quad \frac{{V_1}^2}{2g} = \left(\frac{A_2}{A_1}\right)^2 \frac{{V_2}^2}{2g}$$

$$\frac{{V_1}^2}{2g} \ll \frac{{V_2}^2}{2g} \quad \Rightarrow \quad \left(\frac{A_2}{A_1}\right)^2 \ll 1\text{: } \frac{{V_1}^2}{2g} \text{ can be neglected.} \tag{4.17}$$

- *When can elevation differences be neglected?*

$$|z_2 - z_1| = \Delta z \ll \left| \left(\frac{{V_1}^2}{2g} - \frac{{V_2}^2}{2g} \right) \right| \approx \max \left(\frac{V^2}{2g} \right) : \ \Delta z \text{ can be neglected} \tag{4.18}$$

4.3 Energy Grade Line (EGL)

$$z_{\text{EGL}} = \text{total head} = \frac{V^2}{2g} + \frac{p_{\text{CG}}}{\rho g} + z_{\text{CG}} \tag{4.19}$$

4.4 Hydraulic Grade Line (HGL)

$$z_{\text{HGL}} = \text{piezometric head} = \frac{p_{\text{CG}}}{\rho g} + z_{\text{CG}} = z_{\text{EGL}} - \frac{V^2}{2g} \tag{4.20}$$

If there is no pump in the pipe system:

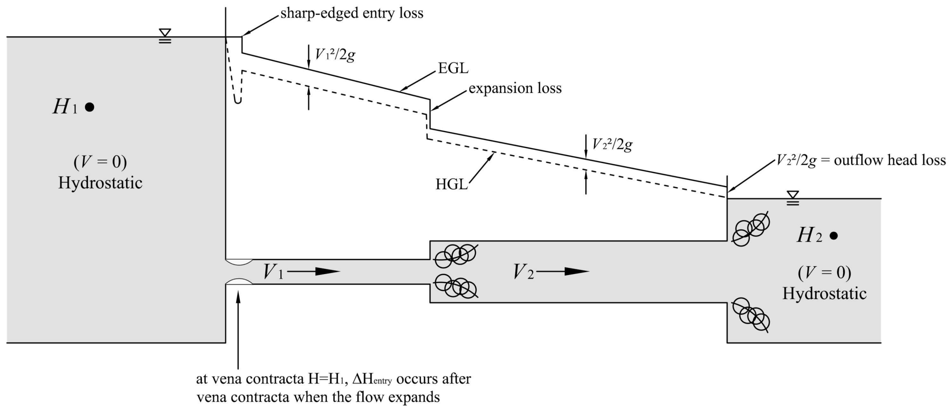
1. z_{EGL} always decreases in the direction of flow ($\Delta H > 0$) or stays constant (in cases when $\Delta H \approx 0$).
2. z_{HGL} is always a distance of $V^2/2g$ (one velocity head) below the EGL. z_{HGL} can increase in the flow direction when $V^2/2g$ decreases due to a flow expansion.

   (See figure on next page.)

4.5 Application of Bernoulli Principle

$$H_1 = \frac{{V_1}^2}{2g} + \frac{p_1}{\rho g} + z_1 = H_2 + \sum_{1 \to 2} \Delta H \tag{4.21}$$

$$= \frac{{V_2}^2}{2g} + \frac{p_2}{\rho g} + z_2 + \sum_{1 \to 2} \Delta H_f + \sum_{1 \to 2} \Delta H_m$$



1. If H_1 and H_2 are known (e.g. free surface elevations in large reservoirs), the head loss between (1) and (2) is known. Expressing all the head loss terms in terms of $V^2/2g = (Q/A)^2/2g$ leads to an equation for V (or Q) that can be solved. Since frictional losses depend on "f" which in turn depends on Re= $V(4R_h)/\nu$, i.e. on the unknown $V = Q/A$, an iterative solution is required (e.g. guessing the value of "f" [0.02 or value for fully rough turbulent flow $f(\epsilon/(4R_h))$] to get a preliminary value V_1, then improving the guess on "f", etc.)

2. If the discharge Q is known, the head loss between (1) and (2) can be evaluated, and the necessary head, e.g. H_1, required to drive the specified flow rate through the system can be obtained.

3. If Q and H_1 and H_2 are given, the pipe system connecting (1) and (2) can be dimensioned.

5 Conclusions

- Always use Conservation of Mass/Conservation of Volume to reduce the number of unknowns, $Q = VA \Rightarrow V_1 = (A_2/A_1)\,V_2$ or $V_2 = (A_1/A_2)\,V_1$, or to calculate velocities, $V = Q/A$, if Q is known.

- If looking for an answer involving forces, always use Conservation of Momentum. Also, use this when forces are known (or may be estimated with confidence) to obtain the relationship between pressures and velocities. Once solved, the Bernoulli Principle can be used to obtain head loss (which must always be $\geq$ zero). Prime example: Short transition of expanding flow.

- If head loss between (1) and (2) is known (or may be estimated with confidence, or expressed in terms of unknowns), use the Bernoulli Principle to get the relationship between the pressures and velocities at (1) and

(2). Once solved, the Momentum Principle can be used to evaluate forces on/from surroundings. Prime examples: short transition of converging flows ($\Delta H \approx 0$); pipe flow analysis ($\Delta H = \Delta H_f + \Delta H_m$).

- Success depends crucially on the choice of CV; in particular, well behaved flow area identification.