LECTURE # 8

1.060 ENGINEERING MECHANICS II

To solve simple fluid mechanics problems we have established:

Conservation of Mass

$$\sum \dot{M}_{in} - \sum \dot{M}_{out} = \frac{\partial M_{cv}}{\partial t}$$

or

$$\sum \rho_{in} Q_{in} - \sum \rho_{out} Q_{out} = \frac{\partial M_{cv}}{\partial t}$$

Conservation of Volume (Continuity)

$$\sum Q_{in} - \sum Q_{out} = \frac{\partial \forall_{cv}}{\partial t}$$

Discharge-Velocity Relationship

$$Q = VA \; ; \; V = Q/A$$

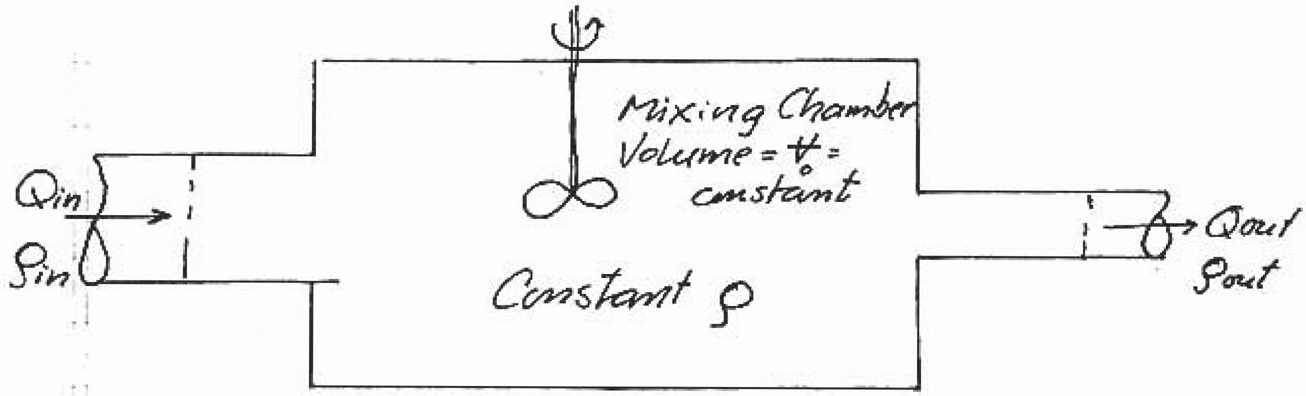
Conservation of Momentum

$$\frac{1}{2}\rho V_s^2 + p_s + \rho g z_s = \text{Constant along } s$$

$$p_h + \rho g z_h + \int_{h_1}^{h_2} \rho \frac{V_s^2}{R} \, dh = \text{Constant along } h \perp s$$

For IDEAL FLUID (inviscid & incompressible)

Example of Mass & Volume Conservation



Bank Account Analogy: Mass

Mass in − Mass out = Change of Mass

$$\sum \rho_{in} Q_{in} - \sum \rho_{out} Q_{out} = \partial M/\partial t = \frac{\partial}{\partial t}(\rho \forall_0) = \forall_0 \frac{\partial \rho}{\partial t}$$

$$\sum \dot{M}_{in} - \sum \dot{M}_{out} = \partial M/\partial t$$

Bank Account Analogy: Volume

$$\sum Q_{in} - \sum Q_{out} = \partial \forall/\partial t = 0 \Rightarrow \sum Q_{in} = \sum Q_{out} = Q$$

Volume in − Volume out = $\partial \forall/\partial t = \partial \forall_0/\partial t = 0$ if incompressible

Combining (single inflow & outflow)

$$\rho_{in} Q_{in} - \rho_{out} Q_{out} = \rho_{in} Q - \rho_{out} Q = \forall_0 \, \partial \rho/\partial t$$

but $\rho_{out} = \rho$, so

$$\rho_{in} Q - \rho Q = \forall_0 \frac{\partial \rho}{\partial t} \Rightarrow \frac{\partial \rho}{\partial t} + \rho \frac{Q}{\forall_0} = \rho_{in}(Q/\forall_0)$$

$Q \cdot T_r = \forall_0$: Time it takes to replace fluid in mixing chamber by "new" fluid coming in = Residence time = time fluid spends in the mixing chamber = $\forall_0/Q$

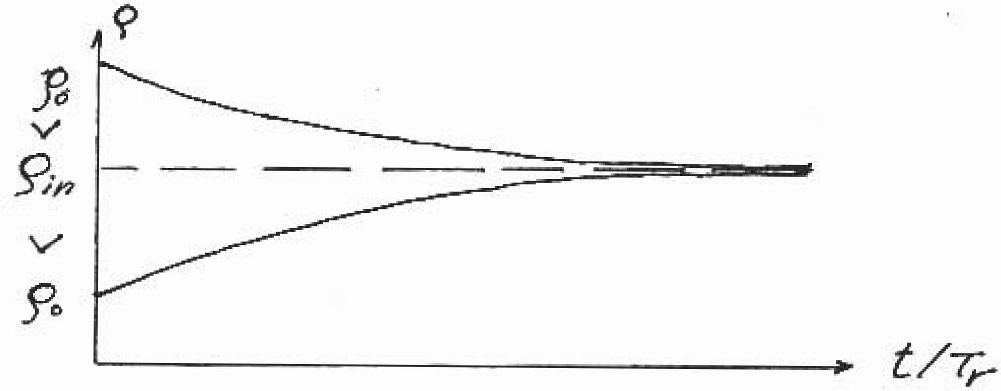
$$\partial \rho/\partial t + \rho/T_r = \rho_{in}/T_r$$

How does ρ vary with time?

Say, $\rho = \rho_o$ @ $t = 0$, and ρ_{in} = constant:
$t \to \infty \Rightarrow \rho$ doesn't change any more - all fluid in mixing chamber replaced by incoming fluid $\rho \to \rho_{in}$

$\rho = C e^{-t/\tau_r} + \rho_{in}$. $\Rightarrow \rho = \rho_o$ @ $t=0 \Rightarrow C = -(\rho_{in} - \rho_o)$

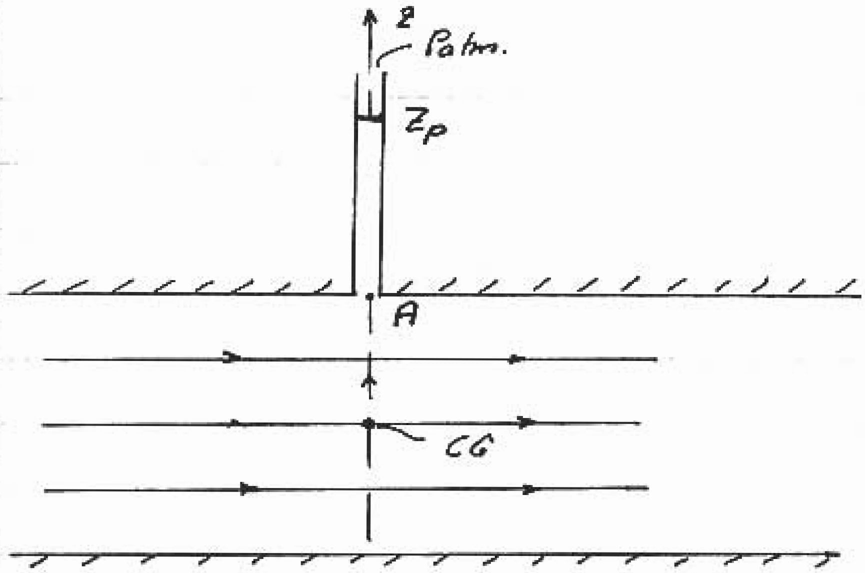
$$\rho = \rho_{in} - (\rho_{in} - \rho_o) e^{-\frac{t}{\tau_r}}$$



Response time scaled by residence time: $\tau_r = \forall_o / Q$
Makes sense: If $\forall_o$ is large, and Q is small, it will take a long time to replace original fluid (ρ_o) by new fluid (ρ_{in})

Note: For an incompressible fluid we can split volume conservation (continuity) - in this case a steady problem $Q_{in} = Q_{out} = Q$ - and mass conservation - in this case an unsteady problem.

WELL BEHAVED FLOW



Far away from changes in flow area

In conduit $h \cdot z$: $p + \rho g z$ = constant along h : $p + \rho g z = p_{CG} + \rho g z_{CG}$ @ CG of A.

In particular, $p_A + \rho g z_A = p_{CG} + \rho g z_{CG}$, at top of conduit. If a standpipe were attached to conduit at A, fluid would flow into standpipe until a pressure was established at A equal to pressure inside conduit at A. The fluid in the standpipe would now be at rest - so $p + \rho g z = p_A + \rho g z_A$ inside standpipe [Hydrostatics]. At top of standpipe $p = p_{atm} = 0$, so the fluid would rise to a level

$0 + \rho g z_p = p_A + \rho g z_A \Rightarrow \underline{z_p - z_A = p_A / \rho g}$

or $\underline{z_p - z_{CG} = p_{CG} / \rho g}$

z_p = Piezometric Head.

For sections where flow is WELLBEHAVED

$V = Q/A$ *is* the velocity for any "h" so if Q and A are known, we know V everywhere across A. We also have that

$p + \rho g z$ is the same everywhere across A

Since any streamline of interest crosses through area A, we have for any streamline that

$$\frac{1}{2}\rho V^2 + p + \rho g z = \text{BERNOULLI CONSTANT}$$

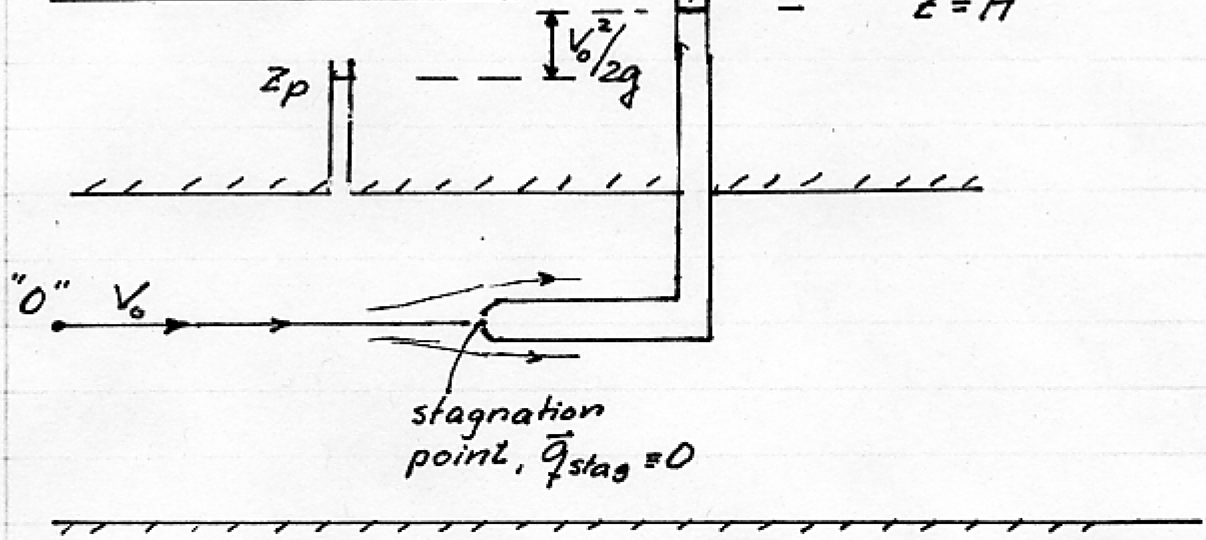
is <u>the same</u>.

Identifying sections where flow is WELLBEHAVED = STAIGHT (NOT CURVED) STREAMLINES is the clue to obtaining the BERNOULLI CONSTANT

Special case: Region of LARGE FLOW AREA
Since $V_0 \sim Q/A_0$, having large A_0 means small V. $\frac{1}{2}\rho V_0^2$ here is much smaller than where $A = A_1$ is smaller, e.g. if $A_1/A_0 \sim 0.1$ then $(V_0/V_1)^2 \sim 0.01$, and $V_0^2 \sim 0.01\, V_1^2$ is negligible $\Rightarrow V \approx 0$. If $V_0 \simeq 0$ we have hydrostatics, so $p_0 + \rho g z_0 = \text{CONST.}$

$$\underline{\frac{1}{2}\rho V^2 + p + \rho g z = 0 + p_0 + \rho g z_0}$$

THE PITOT TUBE



At "0" : $p_0 + \rho g z_0 + \frac{1}{2}\rho V_0^2$

At Stagnation point: $p_{stag} + \rho g z_{stag} + 0$

Along stream line from "0" to Stagnation Point use Bernoulli :

$$p_0 + \rho g z_0 + \frac{1}{2}\rho V_0^2 = p_{stag} + \rho g z_{stag}$$

Inside tube : $\vec{q} = 0 \Rightarrow p + \rho g z = p_{stag} + \rho g z_{stag}$

At free surface in tube : $p + \rho g z = 0 + \rho g H$

Combining we get

$$\rho g H = p_0 + \rho g z_0 + \frac{1}{2}\rho V_0^2$$

or

$$H = \underline{\text{Bernoulli Head}} = \left(\frac{p_0}{\rho g} + z_0\right) + \frac{V_0^2}{2g} =$$

$$\underline{\text{Piezometric Head}} + \underline{\text{Velocity Head}}$$