LECTURE #7

1.060 ENGINEERING MECHANICS II

Combination of Stream Lines and Equi-potential Lines ⇒ Construction of Flow Nets gives us a physical picture of what the flow field, i.e. the VELOCITY FIELD, looks like. It is purely KINEMATIC, and does not tell us anything about the DYNAMICS, i.e. the STRESS FIELD or rather the PRESSURE FIELD associated with the flow we have depicted.

To bring in the DYNAMICS we need to apply NEWTON'S LAW

$$\text{Mass} \times \underline{\text{Acceleration}} = \underline{\text{Force}}$$

or

$$m \cdot \vec{a} = \vec{f}$$

Recall from Lecture #2 that in Eulerian coordinates:

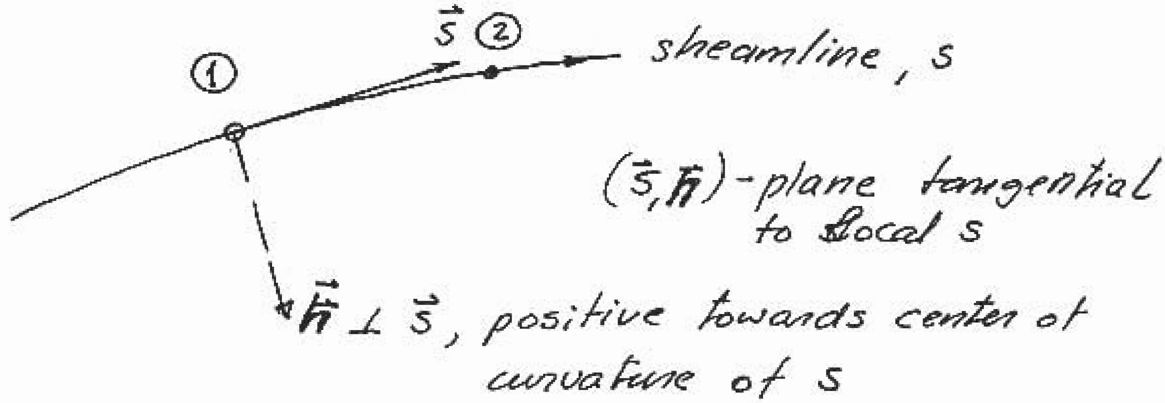
$$\vec{a} = \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q}$$

or, in x-direction,

$$a_x = \frac{\partial u}{\partial t} + (\vec{q} \cdot \nabla) u = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$$

DERIVATION OF THE BERNOULLI EQUATION(S)

Coordinate System



$$\vec{V} = (V_s, V_n) = \{V_s(s,n,t), V_n(s,n,t)\}$$

$$\vec{V}_1 = (V_{s1}, V_{n1}) = (V_1, 0) \quad \text{in } \vec{s},\vec{n} \text{ system}$$

$$\vec{V}_{s2} = V_{s1} + \frac{\partial V_s}{\partial s}\Delta s + \frac{\partial V_s}{\partial n}\Delta n + \frac{\partial V_s}{\partial t}\Delta t \quad \text{(Taylor Expansion)}$$

a_s = fluid particle acceleration in $\vec{s}$-direction =

$$\frac{V_{s2} - V_{s1}}{\Delta t} = \frac{\partial V_s}{\partial t} + \frac{\partial V_s}{\partial s}\underbrace{\frac{\Delta s}{\Delta t}}_{= V_s} + \frac{\partial V_s}{\partial n}\underbrace{\frac{\Delta n}{\Delta t}}_{= V_n = 0 \text{: by def. of } \vec{s}} = \frac{\partial V_s}{\partial t} + \frac{\partial}{\partial s}\left(\frac{1}{2}V_s^2\right)$$

$$\vec{V}_{n2} = V_{n1} + \frac{\partial V_n}{\partial s}\Delta s + \frac{\partial V_n}{\partial n}\Delta n + \frac{\partial V_n}{\partial t}\Delta t \quad \text{(Taylor Expansion)}$$

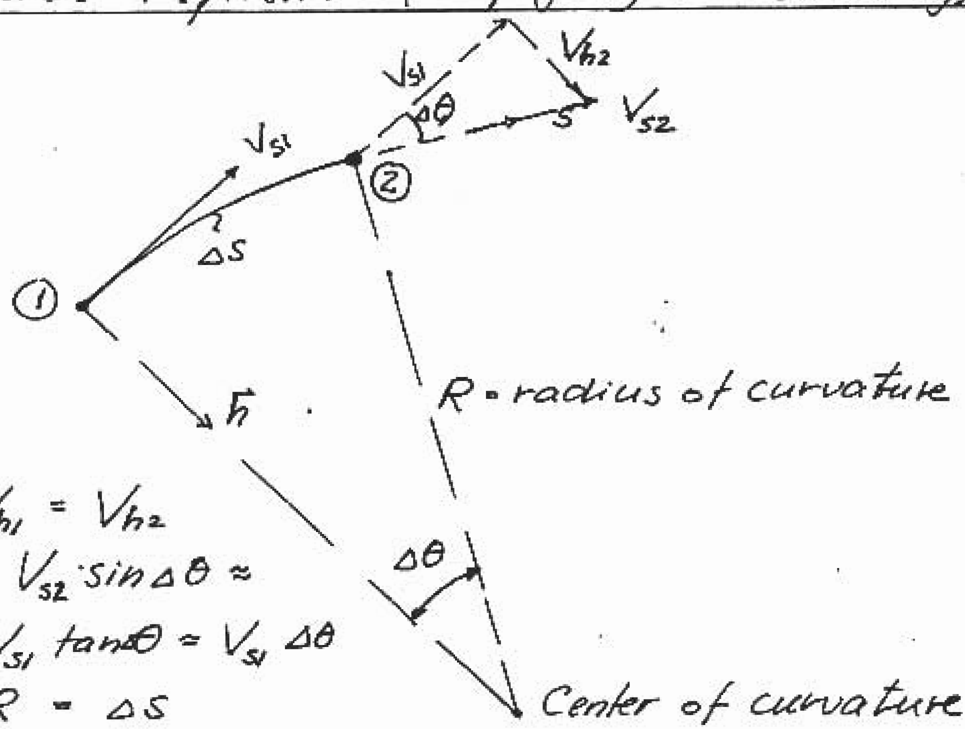
a_n = fluid acceleration in the original $\vec{n}$ direction =

$$\frac{V_{n2} - V_{n1}}{\Delta t} = \frac{\partial V_n}{\partial t} + \frac{\partial V_n}{\partial s}\underbrace{\frac{\Delta s}{\Delta t}}_{= V_s} + \frac{\partial V_n}{\partial n}\underbrace{\frac{\Delta n}{\Delta t}}_{V_n = 0 \text{: by def of } \vec{s}} = \frac{\partial V_n}{\partial t} + \frac{\partial V_n}{\partial s}V_s$$

$$= \frac{\partial V_n}{\partial t} + \frac{V_s^2}{R}$$

R = radius of curvature of s at point (origin) of $(\vec{s}, \vec{n})$

A quick refresher (hopefully unnecessary)



$V_{n2} - V_{n1} = V_{n2}$

$V_{n2} = V_{s2} \sin\Delta\theta \approx$

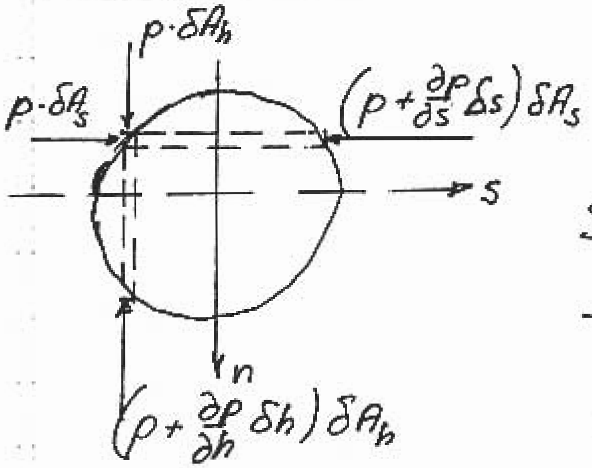
$V_{s1} \tan\Delta\theta = V_{s1}\,\Delta\theta$

$\Delta\theta \cdot R = \Delta s$

So: $$\frac{V_{n2}-V_{n1}}{\Delta s} = \frac{\partial V_n}{\partial s} = \frac{V_{n2}}{\Delta s} = \frac{V_{s1}\,\Delta\theta}{R\,\Delta\theta} = \frac{V_s}{R}$$

and $$V_s \frac{\partial V_n}{\partial s} = V_s^2/R = \text{centripetal acceleration.}$$

Pressure Force



Net force: $-\frac{\partial p}{\partial s} \underbrace{\delta s\,\delta A_s}_{\text{volume of cylinder}}$ in $\vec{s}$

Sum over entire surface:

$$-\sum \frac{\partial p}{\partial s}\delta s\,\delta A_s = -\frac{\partial p}{\partial s}\sum \delta s\,\delta A_s = -\frac{\partial p}{\partial s}\,\delta V$$

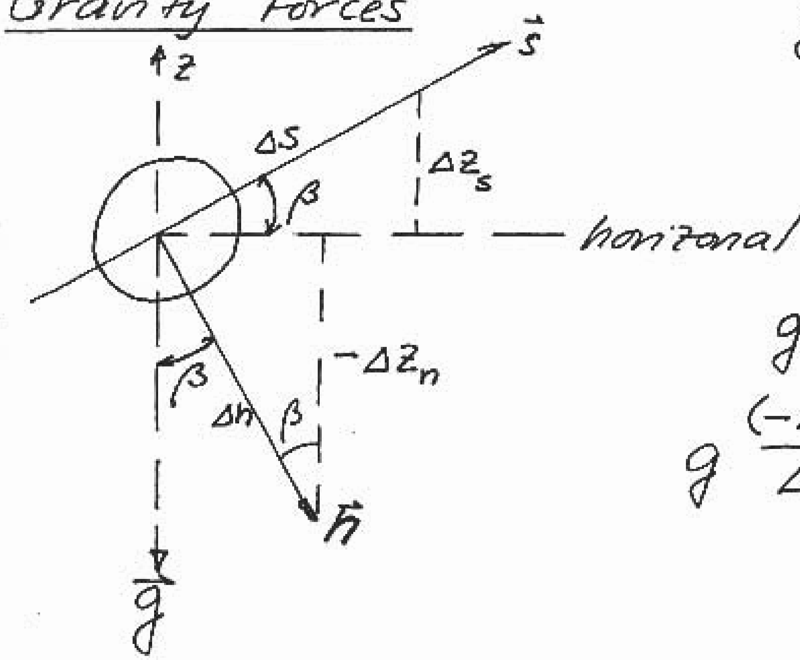
Net force $= -\frac{\partial p}{\partial n}\,\delta n\,\delta A_n$ in $\vec{n}$

Note: Shear stresses are neglected!

Sum over entire surface:

$$-\sum \frac{\partial p}{\partial n}\delta n\,\delta A_n = -\frac{\partial p}{\partial n}\sum \delta n\,\delta A_n = -\frac{\partial p}{\partial n}\,\delta V$$

Gravity Forces



$$g_s = -g\sin\beta = -g\frac{\Delta z_s}{\Delta s} = -g\frac{\partial z_s}{\partial s}$$

$$g_n = g\cos\beta = g\frac{(-\Delta z_h)}{\Delta h} = -g\frac{\partial z_n}{\partial h}$$

Gravity force = mass $\cdot\ \vec{g} = (\rho\,\delta\forall)(g_s, g_h) =$

$$-\rho g\left\{\frac{\partial z_s}{\partial s}, \frac{\partial z_h}{\partial h}\right\}\delta\forall$$

NEWTON'S LAW

Mass × $\overrightarrow{\text{Accel}}$ = $\overrightarrow{\text{Force}}$

$\rho\,\delta\forall \quad \{a_s, a_n\}$ pressure & gravity

In direction of streamline (s)

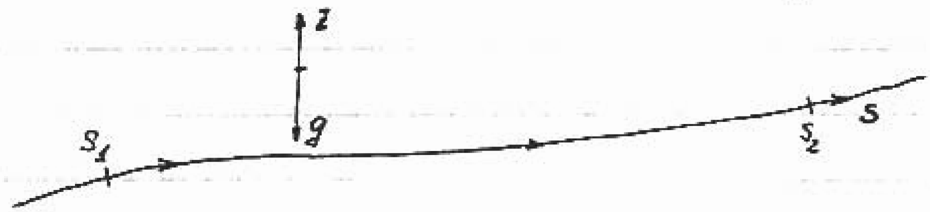
$$(\rho\,\delta\forall)a_s = \rho\left(\frac{\partial V_s}{\partial t} + \frac{\partial}{\partial s}\left(\frac{1}{2}V_s^2\right)\right)\cancel{\delta\forall} - \frac{\partial p}{\partial s}\delta\forall - \frac{\partial z_s}{\partial s}\rho g\,\delta\forall$$

$$\frac{\partial}{\partial s}\left(\frac{1}{2}\rho V_s^2 + p_s + \rho g z_s\right) = -\rho\frac{\partial V_s}{\partial t}$$

In direction ⊥ streamline (h)

$$\frac{\partial}{\partial h}\left(p_h + \rho g z_h\right) = -\rho\frac{\partial V_h}{\partial t} - \rho\frac{V_s^2}{R}$$

BERNOULLI EQUATION (along $\bar{s}$)



$$\frac{\partial}{\partial s}\left[\frac{\rho}{2}V_s^2 + p_s + \rho g z_s\right] = -\rho\frac{\partial V_s}{\partial t}$$

Integrate along streamline from $s=s_1$ to $s=s_2$

$$\left[(\rho/2)V_s^2 + p_s + \rho g z_s\right]_{s_1}^{s_2} = -\int_{s_1}^{s_2}\rho\,\partial V_s/\partial t$$

If flow is steady $\rightarrow \partial/\partial t = 0$ and we have

THE BERNOULLI EQUATION ALONG STREAM LINE

$$\frac{1}{2}\rho V_s^2 + p_s + \rho g z_s = \text{CONSTANT (along } s)$$

1) If $V_s = 0$ everywhere (any line is a streamline)
Bernoulli reduces to Hydrostatics

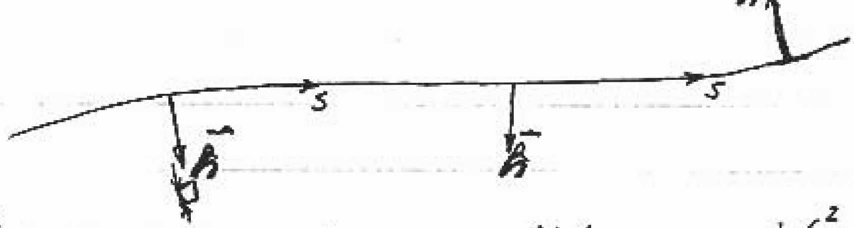
2) If $V_s \neq 0$ Bernoulli gives a relationship between velocity, pressure, and elevation

3) Only 1/3 more difficult than Hydrostatics

$$\frac{1}{2}\rho V_s^2 + p_s + \rho g z_s = \frac{1}{2}\rho V_{s0}^2 + p_{s0} + \rho g z_{s0} = \text{Const.}$$

We need to locate a point, "o", where we know: Pressure (p_{s0}) and Elevation (z_{s0}) (old hat - same as what we needed for Hydrostatics) and now also the Velocity (V_{s0}) at that point. Look for place where flow area is LARGE compared to other regions of the flow. Volume conservation states $V_{s0} A_0 = V_s A_s$: If $A_0 \gg A_s$ then $V_{s0} = (A_s/A_0)V_s \ll V_s$ and we can take $V_{s0} \approx 0$

BERNOULLI $\perp$ STREAMLINES (along $\vec{n}$)



$$\frac{\partial}{\partial n}\left(p_n + \rho g z_n\right) = -\rho \frac{\partial V_n}{\partial t} - \rho \frac{V_s^2}{R}$$

If steady flow $\Rightarrow \partial/\partial t = 0$

If streamlines a <u>straight</u> lines $\Rightarrow R \Rightarrow \infty$

$$p_n + \rho g z_n = \text{CONSTANT} \quad (\perp \vec{s})$$

i.e

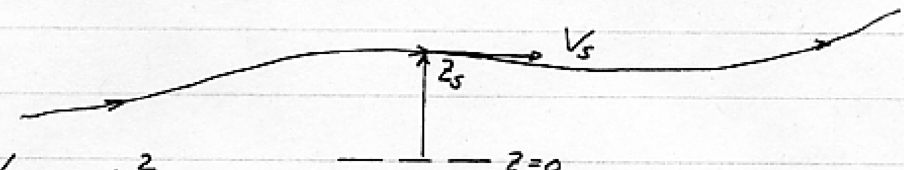
Pressure variation is Hydrostatic normal to straight streamlines!

Hydrostatics is a balance of gravity and pressure forces. When this balance is established $\perp \vec{s}$ there is NO FORCE acting on a fluid particle in direction $\perp \vec{s}$ - hence there is no acceleration of a fluid particle in direction $\perp \vec{s}$. Thus, the particle will move along a straight line in the $\vec{s}$-direction

Conversely, if streamline is "curved", i.e. R not ∞, it moves in an approximately circular path (locally) and this is only possible if there is a net force towards the center of curvature that produces the "centripetal acceleration"

THE BERNOULLI EQUATION(S)

Along a streamline (for steady flow)



$$\frac{1}{2}\rho V_s^2 + p_s + \rho g z_s = \text{CONSTANT} \quad \text{or since } \rho g = \text{const}$$

Bernoulli "Head" $H = V_s^2/2g + p_s/\rho g + z_s = \text{CONSTANT along } \vec{s}$

Most notable LIMITATION: Does not account for shear stresses in the fluid – the fluid is assumed "ideal", i.e. inviscid (and incompressible).

1) When $V_s \equiv 0$ everywhere ⇒ Reduces to HYDROSTATICS

Need location where p, z, and V are known to apply this.

2) When $V_s \equiv 0$ in a region pressure varies HYDROSTATICALLY

Perpendicular to a streamline (for steady flow)

$$p_n + \rho g z_n = -\int_{n_1}^{n_2} \rho \frac{V_s^2}{R}\, dn$$

R = radius of curvature of streamline

1) Pressure is not varying hydrostatically in $\vec{n}$, if R is finite, i.e. streamlines are curved

2) Pressure varies hydrostatically in $\vec{n}$, i.e. $\perp \vec{s}$, if STREAMLINES ARE STRAIGHT LINES

FLOW REGIONS WHERE 2) IS VALID ARE REFERRED TO AS "WELLBEHAVED FLOW SECTIONS"