LECTURE # 32

1.060 ENGINEERING MECHANICS II

MORE GRADUALLY VARIED FLOW EXAMPLES

General Comments:

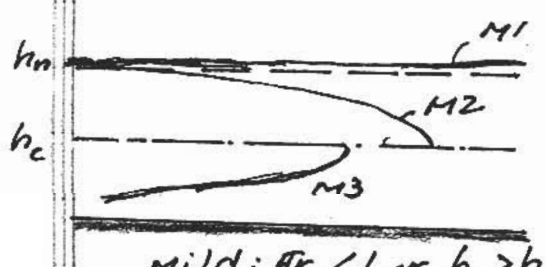
Some basic rules:

R#1: Subcritical flow, $Fr<1$ or $h>h_c$, is always controlled from downstream location.

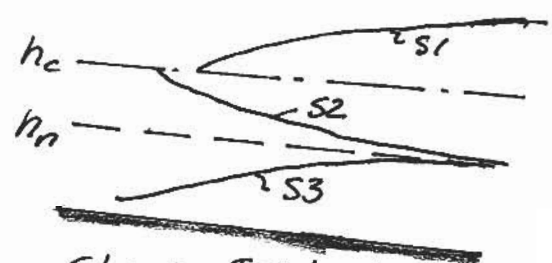
R#2: Supercritical flow, $Fr>1$ or $h<h_c$, is alway having upstream control

R#3: In the absence of any control the only possible flow is normal, $h=h_n$.

R#4: Gradually varied flow MUST follow surface profiles given by M1-3 or S1-3



Mild: $Fr_n<1$ or $h_n>h_c$



Steep: $Fr_n>1$ or $h_n<h_c$

R#5: Transition from a supercritical to a subcritical flow ~~is~~ possible only through a hydraulic jump.

Preliminary considerations

Limiting our channel to be <u>wide</u> and <u>rectangular</u> formulae become simple.

Normal Depth

$$q = \frac{Q}{b} = \frac{1}{n} h_n^{5/3} \sqrt{S_0} \Rightarrow h_n = \left(\frac{nq}{\sqrt{S_0}}\right)^{3/5} \qquad (1)$$

since $R_h = \frac{A}{P} = \frac{bh}{b+2h} = \frac{h}{1+2h/b} \approx h$

Critical Depth

$$Fr = \frac{V}{\sqrt{gh}} = \frac{q}{\sqrt{g}\, h^{3/2}} = 1 \text{ for critical flow } h = h_c$$

$$h_c = \left(q^2/g\right)^{1/3} \qquad (2)$$

E_c = specific energy for $h = h_c = h_c + \frac{V_c^2}{2g} =$

$$h_c + \left(\frac{V_c^2}{g h_c}\right) \frac{h_c}{2} = h_c + 1 \cdot \frac{h_c}{2} = \frac{3}{2} h_c \qquad (3)$$

Hydraulic Jump Condition

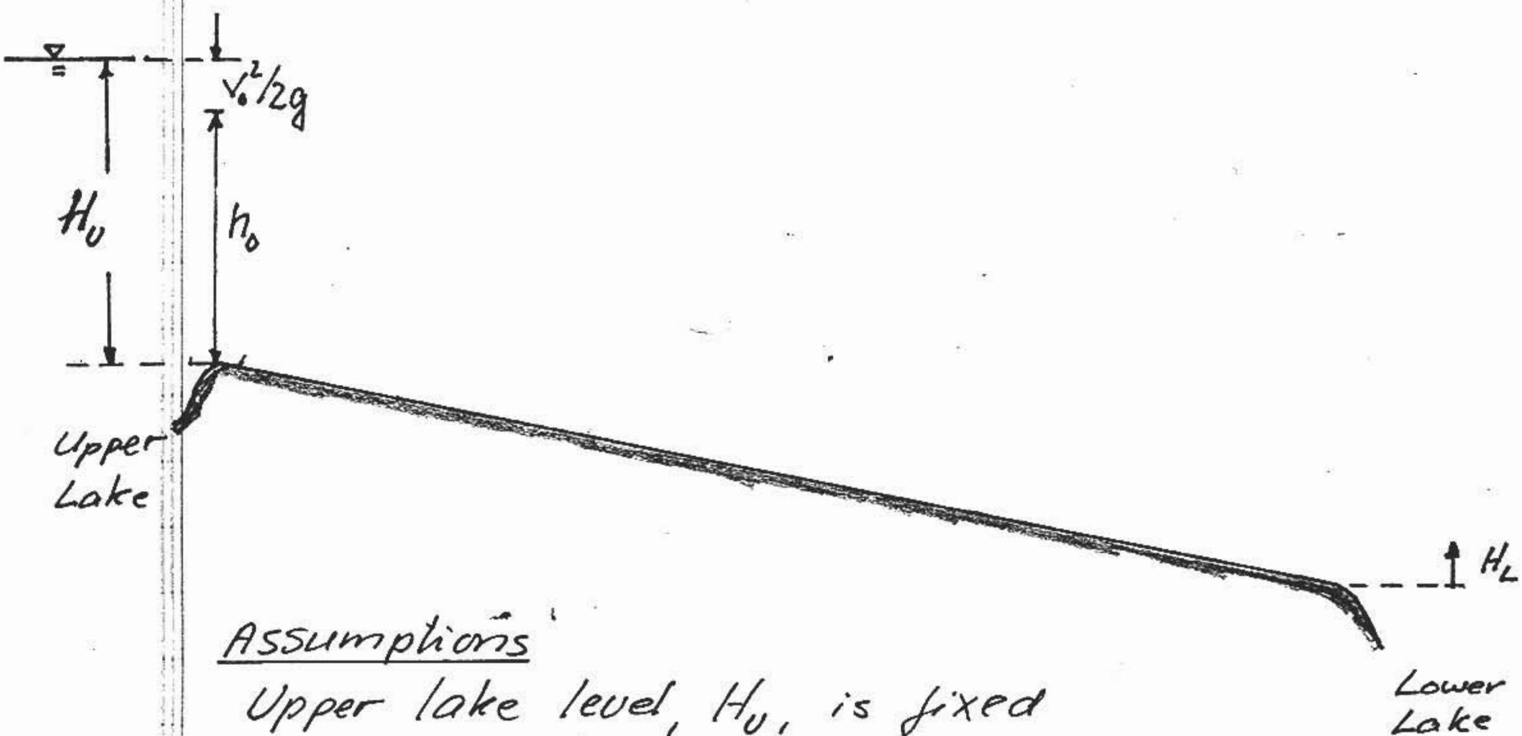
$$h_{conjugate} = \frac{h}{2}\left(-1 + \sqrt{1 + 8 Fr^2}\right) \qquad (4)$$

where

h = known depth ; $Fr^2 = \frac{q^2}{g h^3}$

Upstream flow must be supercritical

THE TWO-LAKE PROBLEM



Assumptions

Upper lake level, H_U, is fixed
Lower lake level, H_L, is variable
Channel is wide rectangular, prismatic, of constant slope S_0, and known Manning's 'n'.
Channel is assumed LONG, so that normal depth can be reached (in most cases)

1 Discharge from Upper Lake into Channel

Flow from lake to channel is a short transition of a converging flow $\Rightarrow \Delta H_{ent} = 0$

$$H_U = h_0 + \frac{V_0^2}{2g}$$

If slope is "steep" flow goes from subcritical (in lake) to supercritical (in "steep" long channel). Flow must pass critical at transition, i.e.

$h_0 = h_c = \frac{2}{3} H_U$ (using (3))

or

$$q = q_c = h_c \cdot V_c = \frac{2}{3} H_U \sqrt{g \frac{2}{3} H_U} = \left(\frac{2}{3}\right)^{3/2} \sqrt{g}\, H_U^{3/2} \quad (5)$$

But is "slope" steep? We don't know until we have found normal depth. However, with $q = q_c$ (assuming "steep" is correct) we have from (1)

$$h_n = \left(\frac{n q_c}{\sqrt{S_0}}\right)^{3/5} \begin{cases} < h_c & \text{slope } \underline{\text{is}} \text{ steep} \\ > h_c & \text{slope } \underline{\text{is not}} \text{ steep = mild} \end{cases} \quad (6)$$

If slope is steep, $h_n < h_c$, $Fr_n > 1$, then flow passes critical depth at entrance to channel and proceeds along an S2-curve until h_n is reached [channel assumed long enough for this to happen] with $q = q_c$.

If slope is not steep it is mild, $h_n > h_c$, $Fr_n < 1$, and does not pass through critical depth at entrance to channel. In the absence (assumed for the moment) of any downstream control, flow must hit normal flow at the entrance, i.e. $h_0 = h_n$, and we have

$$H_U = h_n + \frac{V_n^2}{2g} = h_n + \frac{q_n^2}{2g h_n^2}$$

but since flow is normal we also have (from (1))

$$q_n = \frac{1}{n} h_n^{5/3} \sqrt{S_0} \Rightarrow \frac{q_n^2}{2g h_n^2} = \frac{S_0}{n^2} \frac{h_n^{10/3}}{2g h_n^2} = \frac{S_0}{2g n^2} h_n^{4/3}$$

Substitution now gives

$$H_U = h_n + \frac{S_0}{2n^2 g} h_n^{4/3} = h_n \left(1 + \frac{S_0}{2n^2 g} h_n^{1/3}\right)$$

or

$$h_n = \frac{H_U}{\left(1 + \frac{S_0}{2gn^2} h_n^{1/3}\right)}. \qquad (7)$$

from which h_n is readily obtained by iteration.
With h_n known we have

$$q = q_n = V_n h_n = \frac{1}{n} h_n^{5/3} \sqrt{S_0} \qquad (8)$$

Just to make sure, the faint at heart, may want to check if slope now is mild by evaluating and showing

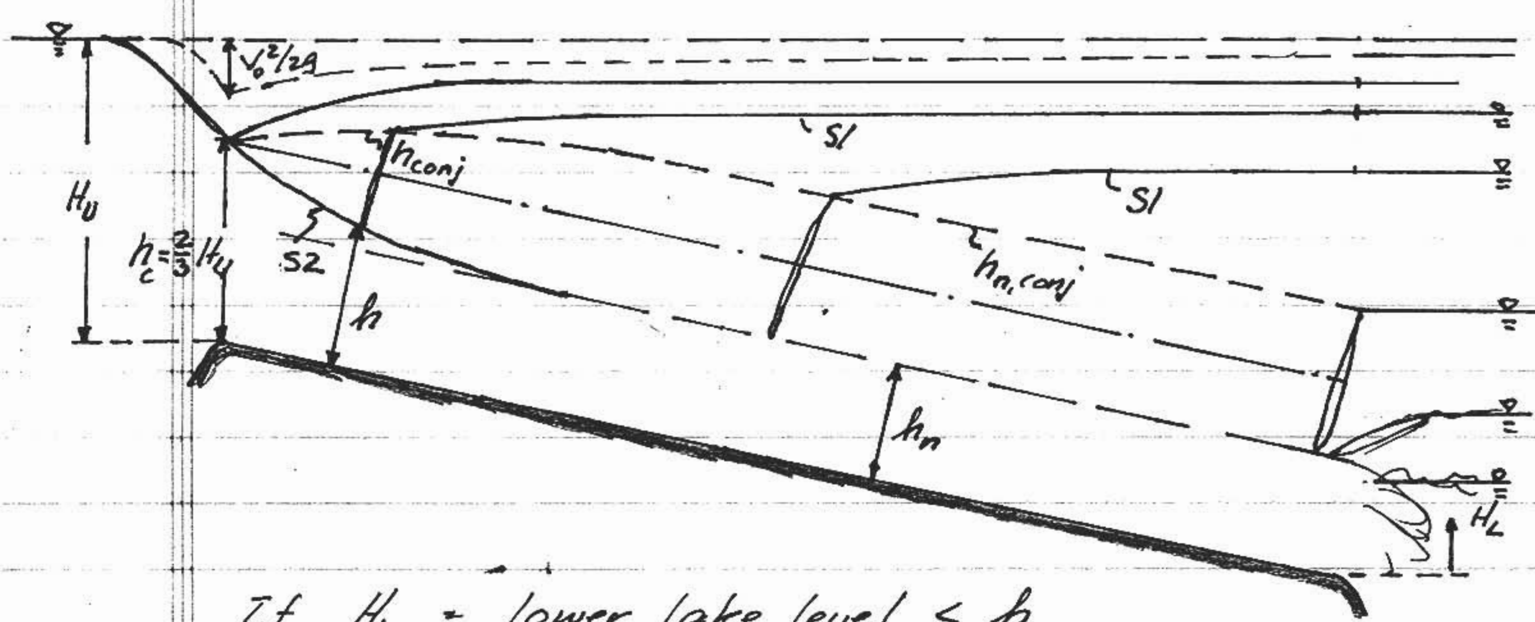
$$Fr_n = \frac{q_n^2}{g h_n^3} < 1$$

but this is really not necessary since we know that q is maximum for a given specific energy, H_U, when flow is critical, $q = q_c$. For any other condition than $q = q_c$ we therefore know that $q = q_n < q_c$, but $h_n > h_c$ since inflow is subcritical, i.e. $Fr_n^2 = (q_c^2/gh_c^3)(q_n/q_c)^2(h_c/h_n)^3 = (q_n/q_c)^2(h_c/h_n)^3 < 1$.

Finally, we have for $q = q_n$ the corresponding critical depth for the mild slope case (from (2))

$$h_{c,mild} = \left(\frac{q_n^2}{g}\right)^{1/3} \qquad (9)$$

STEEP SLOPE PROFILES



If H_L = lower lake level < h_n
the flow is supercritical all the way. It reaches h_n through an S2-profile and since $h_n < h_c$ or $Fr_n > 1$ the flow can not respond to anything happening downstream, i.e. it remains at h_n until it discharges in a jet-like fashion into the lower lake.

The only way the flow in the channel is affected by the downstream (lower) lake level is if the channel flow somehow becomes sub-critical (downstream control). Since normal flow is reached and is supercritical, the only way to get to a subcritical flow in the channel is through a HYDRAULIC JUMP. To have a hydraulic jump the depth downstream of the jump must be conjugate to the upstream depth, i.e. given by (4). The conjugate depth to the depth along the S2-profile ($h > h_n$ but $h < h_c$)

is shown in sketch. At any point along the channel the flow can jump from its upstream supercritical value to its conjugate (subcritical) depth. The first time this becomes a possibility is when the lower lake level, H_L, is equal to $h_{n,conj}$. Thus,

$\underline{H_L = h_{n,conj}}$ Flow jumps from h_n to $h_{n,conj}$ just before it exits channel into lower lake

For $\underline{h_n < H_L < h_{n,conj}}$ a partial jump starts in the channel and extends into the lower lake.

Now, for $H_L > h_{n,conj}$ the lower lake level is so high that the flow is backed up into the steep channel, i.e. profile starts at $H_L = h_L$ at the outflow and follows a S1-profile up into the channel. Since the depth is $> h_{n,conj} > h_c$ the flow is subcritical (and that is why it can feel what is going on downstream!). When the S1-profile's depth has decreased until it reaches a value of $h_{n,conj}$ a jump takes place. As H_L increases this jump moves farther and farther up the channel, until the S1-profile just manages to reach h_c at the entrance from the upper lake. This value of $H_L = H_{L,lim}$ signals a change in outflow conditions from the upper lake and therefore q will no longer be q_c since the flow no longer can get down to $\frac{2}{3}H_0$ at the channel entrance

$h_{n,conj} < H_L < H_{L,lim}$ Flow enters channel passing through critical depth, follows an S2-profile until it makes a jump to its conjugate depth and proceeds along an S1-profile until reaching the lower lake. As H_L increases, the jump forms closer to the upper lake and for $H_L = H_{L,lim}$ there is no longer a hydraulic jump in the channel since the S1-profile starts at $h = h_c$ at the entrance.

Note, when $H_L = H_{L,lim}$ the flow is subcritical along the entire length of the channel, except right at the entrance where it is right at critical. Thus, any further increase in H_L will make the flow subcritical everywhere including right at the entrance to the channel.

For $H_L > H_{L,lim}$ $q \neq q_c$, so $q < q_c$, but it is not normal flow as in the case of a mild sloping channel. What do we do?

$$H_U = h_o + \frac{q^2}{2gh_o^2}$$

still holds with h_o = depth at start of channel and we know that $h_o > \frac{2}{3}H_U$. Thus we can pick a value of h_o such that

$$\frac{2}{3}H_U < h_o < H_U$$

and for this h_o the corresponding value of q is obtained.

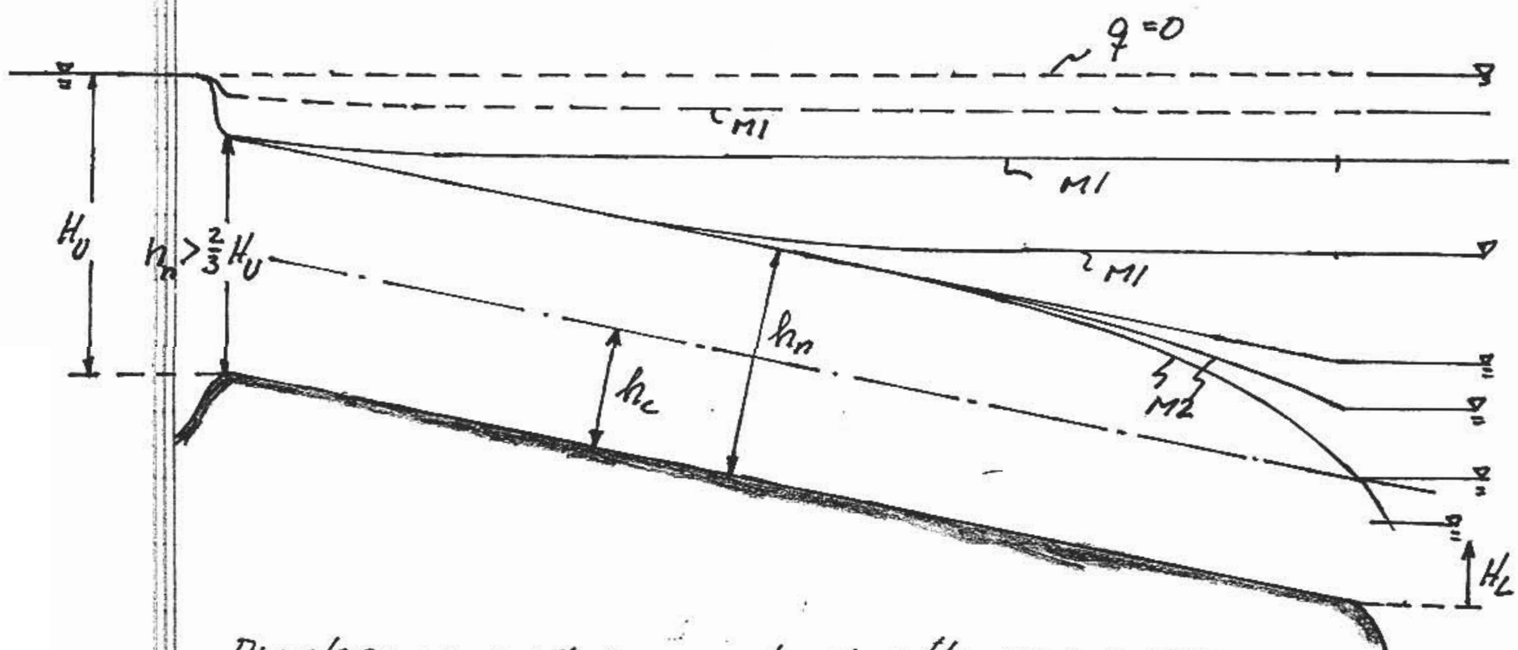
With this value of q and starting from h_o the surface profile in the channel (S1-profile) can be calculated and the value of h obtained at the outflow of the channel into the lower lake must be the level in the lake, H_L.

Obviously, if $h_o = H_U$ there will be no flow into the channel, i.e. $q = 0$, and the lower lake will be at a level equal to that of the upper lake

Thus $H_{L,lim} < H_L < Z_L = Z_U \Rightarrow q_c < q < 0$. and

$$\tfrac{2}{3} H_U < h_o < H_U.$$

MILD SLOPE PROFILES



Discharge and normal depth are given by (7) and (8), and h_n is hit immediately at start of channel and is maintained until influenced by conditions imposed by the lower lake (flow is subcritical $h_n > h_c$ so there is downstream control!).

$\underline{H_L < h_c}$: Flow is drawn down in channel from h_n to critical depth at the outflow from the channel to the lower lake. (Free outflow over a "brink"). Follows an M2-profile since channel is assumed long enough to reach h_n.

$\underline{h_c < H_L < h_n}$: Flow is drawn down through an M2-profile to meet level in lower lake

$\underline{h_n < H_L < H_{L,lim}}$: Flow is backed up into the channel. Meets H_L at lower lake and h_n upstream. But when $H_L = H_{L,lim}$ the lower lake level is so high that h_n is not reached until right at the entrance from the upper lake. $q = q_n$ still holds

$\underline{H_L > H_{L,lim}}$: h_0 = depth at entrance $> h_n$. Discharge changes! Solution is as for steep channel when $H_L > H_{L,lim}$, i.e.

$$H_U = h_0 + \frac{q^2}{2gh_0^2} \qquad (h_n < h_0 < H_U)$$

Pick h_0, get q. With this q and starting from h_0 surface profile is computed and depth a outflow to lower lake meets (and defines) $H_L > H_{L,lim}$. When $h_0 = H_U$, $q = 0$ and there is no flow between the lakes – free surface is horizontal everywhere.