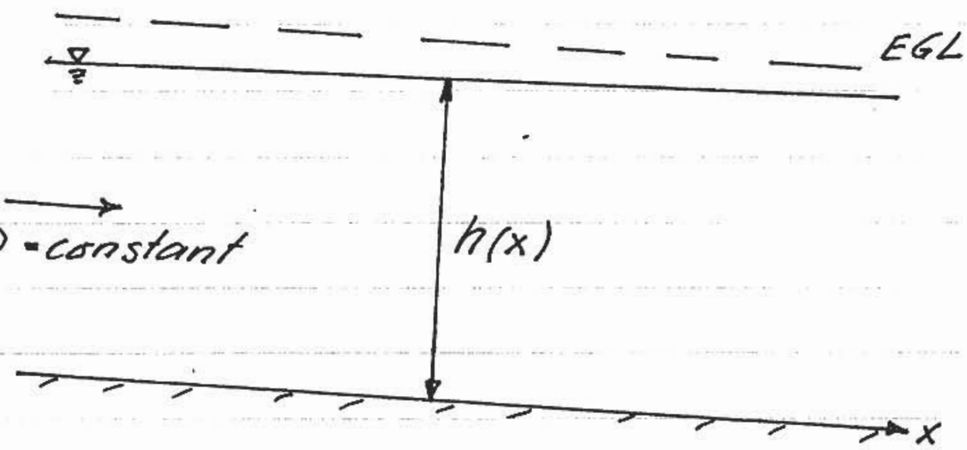
LECTURE # 30

1.060 ENGINEERING MECHANICS II

GRADUALLY VARIED FLOW



$$\frac{dh}{dx} = \frac{S_o - S_f}{1 - Fr^2}$$

$S_o = \sin\beta$ = bottom slope

$Fr^2 = \dfrac{Q^2 b_s}{g A^3}$ = Froude # squared

$S_f = -\dfrac{\partial H}{dx}$ = slope of EGL

$$S_f = \frac{\tau_s}{\rho g A/P} = \frac{f}{8g}\frac{Q^2}{A^3/P} = \frac{n^2 Q^2}{A^{10/3}/P^{4/3}}$$

Normal Flow: $S_o = S_f$ gives h_n = Normal Depth

If $h \gtrless h_n$ then $S_f \lessgtr S_o$ or $S_o - S_f \gtrless 0$

Since $\begin{Bmatrix}\text{larger}\\ \text{smaller}\end{Bmatrix}$ h gives $\begin{Bmatrix}\text{smaller}\\ \text{larger}\end{Bmatrix}$ V and therefore $\begin{Bmatrix}\text{smaller}\\ \text{larger}\end{Bmatrix}$ τ_s & S_f

<u>Critical Flow</u> : $Fr^2 = \frac{Q^2 b_s}{g A^3} = \frac{V^2}{g h_m} = 1$ gives $h = h_c =$ Critical Depth.

<u>If $h \gtrless h_c$ then $Fr^2 \lessgtr 1$ or $1 - Fr^2 \gtrless 0$</u>

since $\begin{Bmatrix} \text{larger} \\ \text{smaller} \end{Bmatrix}$ h gives $\begin{Bmatrix} \text{larger} \\ \text{smaller} \end{Bmatrix}$ $\frac{A^3}{b_s}$ and hence $\begin{Bmatrix} \text{smaller} \\ \text{larger} \end{Bmatrix}$ Fr^2

Thus, we can <u>predict</u> with <u>certainty</u> the <u>nature</u> of the depth variation in the direction of flow

$$h \begin{Bmatrix} \text{increases} \\ \text{decreases} \end{Bmatrix} \text{ with } x, \text{ if } \frac{dh}{dx} \gtrless 0$$

by simply considering the <u>signs</u> of the numerator, $(S_o - S_f)$, and denominator, $(1 - Fr^2)$, in the gradually varied flow equation, and these signs depend, in turn, on the local depth, h, relative to normal depth, h_n, and critical depth, h_c, in the given channel (prismatic) for the given discharge, Q.

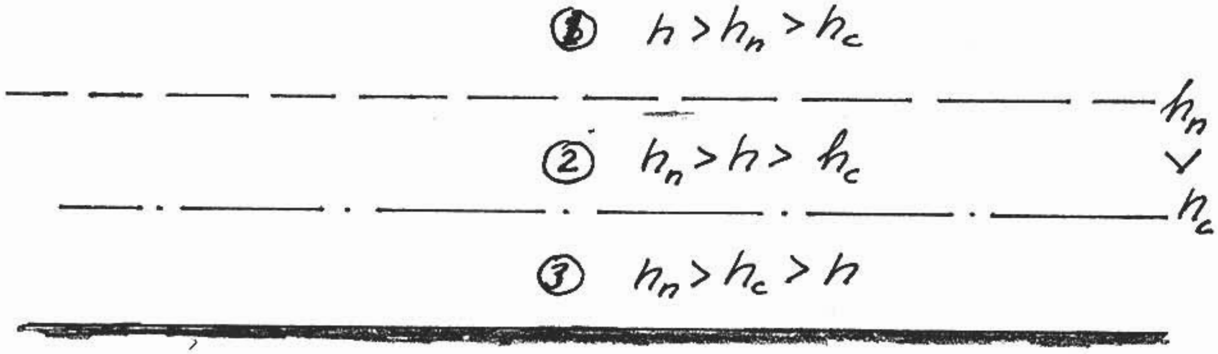
<u>IT DOES NOT GET ANY BETTER THAN THIS!</u>

All that is needed is the relative magnitude of h_n and h_c, and this is related to the slope of the channel, S_o.

If normal flow is $\begin{Bmatrix} \text{subcritical} \\ \text{supercritical} \end{Bmatrix}$ then $Fr_n \lessgtr 1$ and the slope, S_o, is referred to as $\begin{Bmatrix} \text{MILD} \\ \text{STEEP} \end{Bmatrix}$

GRADUALLY VARIED FLOW PROFILES

Mild Slope: $Fr_n < 1 \Rightarrow h_n > h_c$



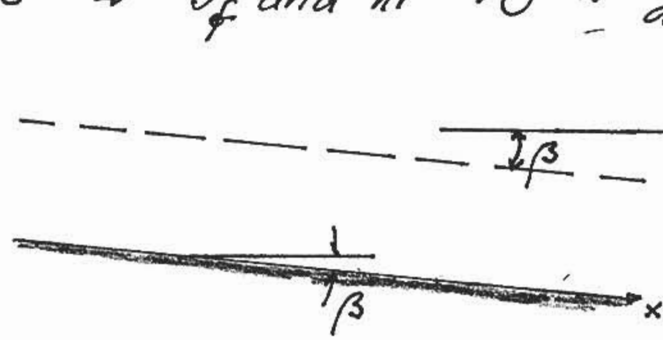
M1 Profile – $h > h_n > h_c$

$$\frac{dh}{dx} = \frac{S_o - S_f}{1 - Fr^2} = \frac{+}{+} > 0$$

M1 Profiles must have a depth that increases in the direction of flow – or conversely – a depth that decreases in the upstream direction.

Limiting behavior:

$$h \to \infty \Rightarrow S_f \text{ and } Fr^2 \to 0 \Rightarrow \frac{dh}{dx} \to S_o = \sin\beta$$



Free surface becomes horizontal as $h \to$ "∞".

Since $h \to \infty$ gives $V \to 0$ conditions approach hydrostatics: Hence a horizontal free surface.

$$h \to h_n \Rightarrow S_f \to S_0 \,, \quad Fr^2 \to Fr_n^2 < 1 \Rightarrow \frac{dh}{dx} \to 0$$

Depth approaches normal depth - which is an obvious result - but how does it approach "normalcy"?

To answer this, we consider a simple channel of the "very wide, rectangular" variety. For this case we have

$$\frac{dh}{dx} = \frac{S_0}{1-Fr_n^2}\left(1 - \frac{S_f}{S_0}\right) = \frac{S_0}{1-Fr_n^2}\left(1 - \left(\frac{h_n}{h}\right)^3\right)$$

when S_f is expressed by the Chezy ~~Manning's~~ formula. Now, since we are looking for the behavior as $h \to h_n$ from above, we take

$$h = h_n + \delta_n(x) \quad \text{with} \quad \delta_n(x) \ll h_n$$

For this the equation may be written

$$\frac{d\delta_n}{dx} = \frac{S_0}{1-Fr_n^2}\left(1 - \left(\frac{h_n}{h_n+\delta_n}\right)^3\right) = \frac{S_0}{1-Fr_n^2}\left(1 - \left(1+\frac{\delta_n}{h_n}\right)^{-3}\right)$$

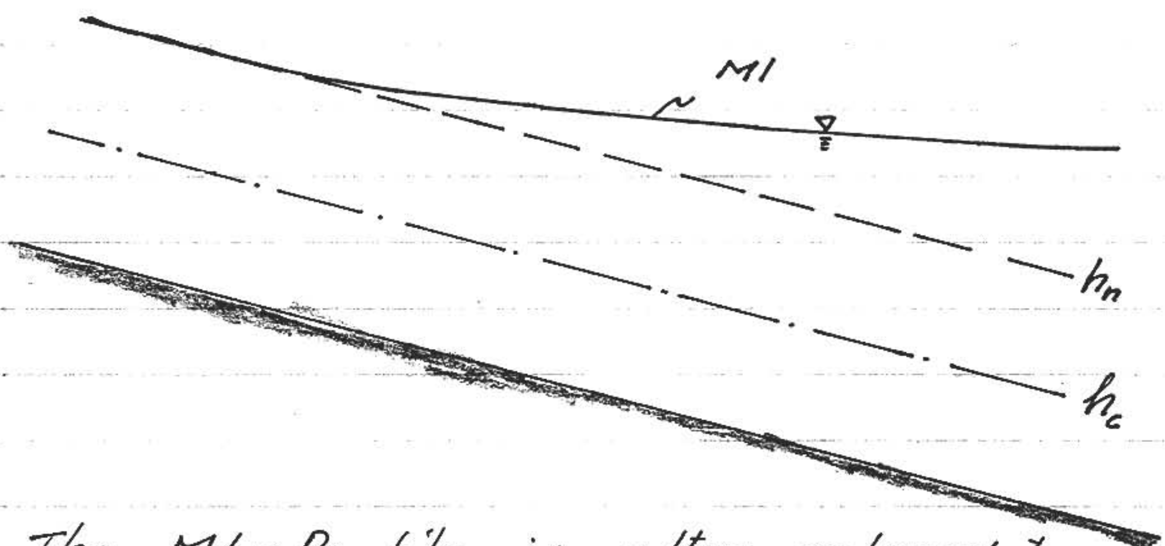
Finally, $\left(1+\delta_n/h_n\right)^{-3} \approx 1 - 3\frac{\delta_n}{h_n}$ when $\frac{\delta_n}{h_n} \ll 1$, and the behavior of δ_n, i.e. the manner in which h_n is approached is given by

$$\frac{d\delta_n}{dx} - \frac{3}{h_n}\frac{S_0}{1-Fr_n^2}\delta_n = \frac{d\delta_n}{dx} - \alpha_n\delta_n = 0$$

The solution to this equation is an exponential variation

$$\delta = \delta_0 e^{\alpha x}$$

where δ_0 = value of δ at "$x=0$". Thus, $\delta \to 0$, i.e. $h \to h_n$, as $x \to -\infty$. So, normal depth is approached "asymptotically" [gets close, but "never" get there] as we go far upstream (x positive in downstream direction).



The M1-Profile is often referred to as a "backwater profile" or "backwater curve" since it is encountered when a flow obstruction such as an underflow gate or a dam "backs up" the water into the mildly sloping channel. Notice, it is a <u>downstream</u> control or condition that imposes a requirement of a depth $h > h_n$ and this control is propagated and felt <u>upstream</u> of control.

This is, of course, related to the fact that the flow along the entire backwater profile has $h > h_n > h_c$ and therefore ~~a~~ subcritical flow which is controlled by downstream conditions.

<u>M2 - Profile</u> - <u>$h_n > h > h_c$</u>

$$\frac{dh}{dx} = \frac{S_0 - S_f}{1 - Fr^2} = \frac{-}{+} < 0$$

M2-profiles must have a depth that decreases in the downstream (increases in the upstream) direction. For this reason, M2-profiles are often referred to as "draw-down profiles" or "draw-down curves".

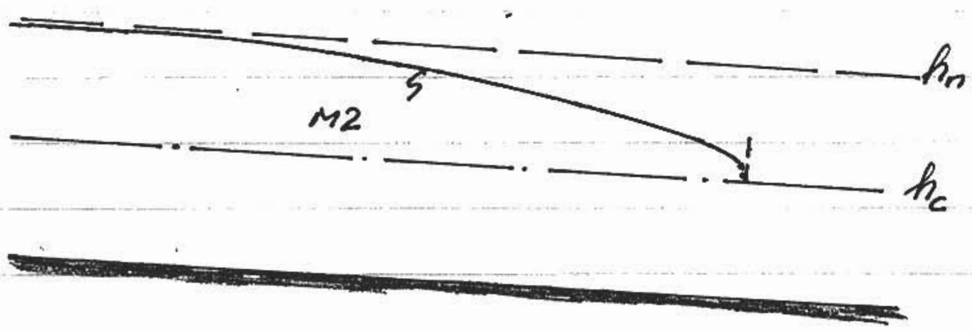
<u>Limiting behavior</u>

$h \to h_n$ behaves in the same manner as obtained for the M1-profile in this limit, i.e. h approaches h_n assymptotically as we go <u>upstream</u> ($h > h_c$ flow is subcritical and has <u>downstream control</u>).

$$h \to h_c \Rightarrow S_0 - S_f < 0 \text{ , } Fr \to 1 \Rightarrow 1 - Fr^2 \to 0 \Rightarrow \frac{dh}{dx} \to -\infty$$

So, critical flow is approached very abruptly! In fact, $dh/dx \to -\infty$ as $h \to h_c$

suggest that the free surface is perpen-dicular to the bottom for $h = h_c$. Even if the bottom slope is small, this means that the free surface is "leaning" forward — an absurd result.



First, we recall that the equation gover-ning gradually varied flow profiles was derived under the assumption of well behaved flow everywhere. This implies that the pressure distribution is hydrostatic and that stream-lines are straight and parallel (nearly). Near critical flow neither of these assumptions hold, i.e. assumptions are violated and solution is invalid! How serious are the consequences of this violation?

To answer this question, we again simpli-fy our analysis by assuming a wide rectan-gular channel and Chezy's formula for flow resistance and examine the behavior of h as h_c is approached by taking

$$h = h_c + \delta_c(x) \quad ; \text{with } \delta_c \ll h_c$$

With these simplifications we have

$$\frac{dh}{dx} = \frac{d\delta_c}{dx} = \frac{S_0 - S_f}{1 - Fr^2} = \frac{S_0\left(1 - S_f/S_0\right)}{1 - \frac{Q^2/b^2}{g\,h^3}} = \frac{S_0\left(1 - \left(\frac{h_n}{h}\right)^3\right)}{1 - \left(h_c/h\right)^3}$$

In the numerator $h = h_c$ whereas $h = h_c + \delta_c$ in the denominator. Thus, we obtain

$$\frac{d\delta_c}{dx} = - \frac{S_0\left(h_n^3 - h_c^3\right)}{h_c^3\left(1 - \left(1 + \frac{\delta_c}{h_c}\right)^{-3}\right)} \approx - \frac{S_0\left(h_n^3 - h_c^3\right)}{h_c^2 \cdot 3\,\delta_c}$$

or

$$2\delta_c \frac{d\delta_c}{dx} = \frac{d(\delta_c)^2}{dx} = -\frac{2}{3}\,\frac{S_0\left(h_n^3 - h_c^3\right)}{h_c^2} = -\alpha_c$$

Therefore,

$$\delta_c = \left(-\alpha_c x\right)^{1/2} = \sqrt{\alpha_c}\,\sqrt{-x}$$

valid for conditions upstream of critical flow conditions, i.e. for $x < 0$.

By differentiation by x we obtain the free surface slope relative to the bottom

$$-\frac{d\delta_c}{dx} = +\frac{1}{2}\sqrt{\alpha_c} \cdot \frac{1}{\sqrt{-x}} > 0$$

and requiring this to be "small", say $< \sqrt{S_0}$, we obtain

$$\frac{1}{2}\sqrt{\alpha_c / S_0} < \sqrt{-x}\,;$$

or

$$-x > \frac{1}{4}\frac{\alpha_c}{S_o} = \frac{1}{6}\left[\left(\frac{h_n}{h_c}\right)^3 - 1\right] h_c = \frac{1}{6}\ \frac{1 - Fr_n^2}{Fr_n^2}\ h_c$$

Hence, even for a very mild slope, corresponding to $Fr_n^2 = O(10^{-2})$, we only have to go a distance of $\approx 16 h_c$ upstream of the location of critical flow to render our assumption of nearly well behaved flow valid. In the context of gradually varied flow profiles this distance of $\approx 16 h_c$ is "nothing" and we need not be concerned by our violation of our assumptions. However, we should never forget that our solution is more conceptual than real as we get close to h_c

M3 - Profile - $h_n > h_c > h$

$$\frac{dh}{dx} = \frac{S_o - S_f}{1 - Fr^2} = \frac{-}{-} > 0$$

For an imposed depth $h < h_c < h_n$ the depth must increase in the direction of flow.

Limiting behavior:

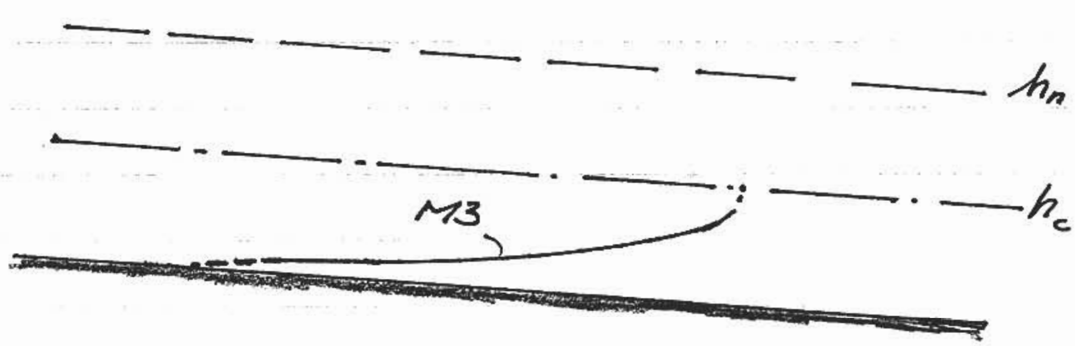
$h \to h_c \Rightarrow \frac{dh}{dx} \to +\infty$ since $1 - Fr^2 \to 0$ and the behavior is analogous to the one we analyzed as $h \to h_c$ for M2-profiles

$$h \to 0 \Rightarrow S_f \to \infty \ \& \ Fr^2 \to \infty \Rightarrow \frac{dh}{dx} \to \frac{S_f}{Fr^2}$$

Again, a wide rectangular channel and Chezy's resistance gives

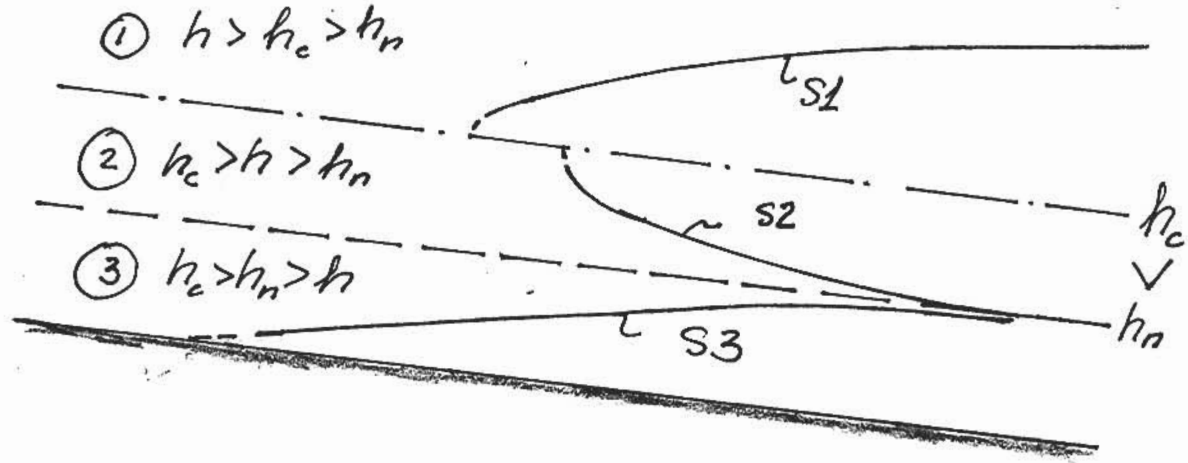
$$\frac{dh}{dx} \to \frac{(\dot{Q}/b)^2 h^{-3} C^{-2}}{(\dot{Q}/b)^2 h^{-3} g^{-1}} = \frac{g}{C^2} = \frac{f}{8}$$

i.e. for very small values of the imposed depth [a depth of zero is obviously non-sense, since finite Q would give infinite velocity!] the depth increases nearly linearly in the downstream direction at a rate governed by the channel's frictional characteristics (e.g. the bottom roughness).



The M3-profile does not, to my knowledge, have a more descriptive "name" like M1 = backwater, M2 = draw-down. It is generally encountered when a flow enters a channel after passing under an underflow gate.

Steep Slope : $Fr_n > 1 \Rightarrow h_c > h_n$



S1 - profile - Backwater Curve - $h > h_c > h_n$

$$\frac{dh}{dx} = \frac{S_o - S_f}{1 - Fr^2} = \frac{+}{+} > 0$$

Depth increases in downstream (decreases in upstream) direction. Encountered when a flow obstruction (dam or underflow gate) back up the water into a steep channel. Notice that even though channel is steep, the flow is everywhere subcritical along the S1 profile. That's why an S1-profile has a downstream control. The limiting behavior is similar to mild slope profiles, i.e. $h \to \infty \Rightarrow dh/dx \to S_o \Rightarrow$ free surface approaches horizontal; $h \to h_c \Rightarrow dh/dx \to +\infty$ i.e. unreliable detail as $h \to h_c$.

S2 - profile - Draw-down Curve - $h_c > h > h_n$

$$\frac{dh}{dx} = \frac{S_o - S_f}{1 - Fr^2} = \frac{+}{-} < 0$$

The S2-profile has a depth that decreases in the direction of flow. It starts out, for $h \to h_c$, with $dh/dx \to -\infty$ and ends, for $h \to h_n$, by approaching normal depth assymptotically. Notice that, since $h < h_c$, the flow is supercritical and the depth is decreasing not so much because it is "drawn down" but more because it started out too high and wants to "get down" to its normal flow value. This situation may occur when a flow exits from under an underflow gate and enters a very steep channel.

S3-profile - "no name" - $h_c > h_n > h$

$$\frac{dh}{dx} = \frac{S_o - S_f}{1 - Fr^2} = \frac{-}{-} > 0$$

Depth increases in direction of flow. Behavior as $h \to 0$ same as M3-profile, and h_n is approached assymptotically in downstream direction. Again, the flow under a gate may result is a S3-profile downstream of the outflow.

"Symbolic" S-profiles are shown on preceding page.

SUMMARY OF GRADUALLY VARIED FLOW PROFILES

NDL = normal depth line ($y_n = h_n$)
CDL = critical depth line ($y_c = h_c$)

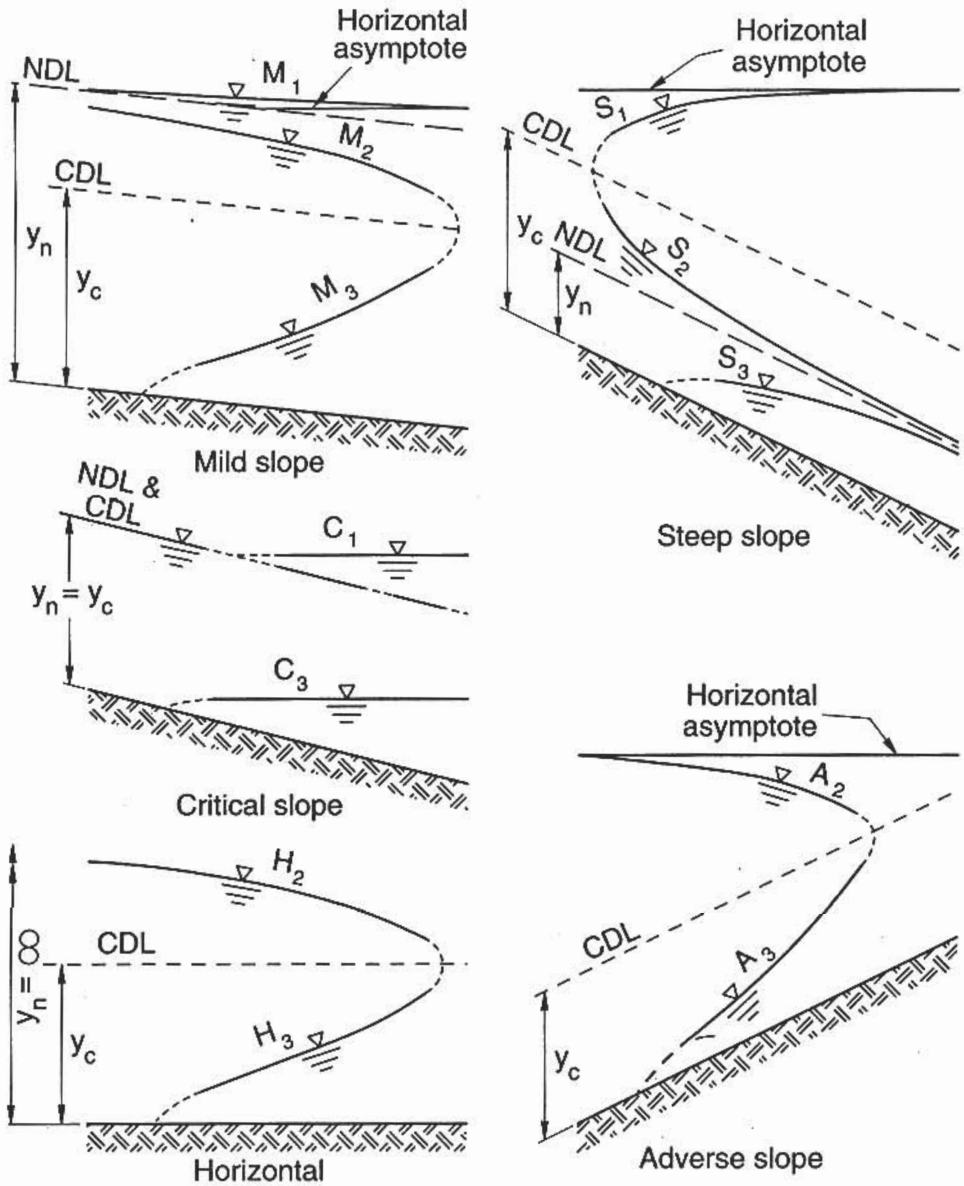


Figure 4-2. Water-surface profiles in gradually varied flow.

(from Jain, "Open-channel Flow", Wiley, NY, (2001))

ESTIMATE OF CHANNEL LENGTH AFFECTED BY GRADUALLY VARIED FLOW

Making certain simplifying assumption regarding channel cross-section variables solutions for gradually varied flow profiles in prismatic channels may be obtained from tabulated values of the gradually varied flow function. Alternatively, the governing differential equation may be solved numerically (without the restriction of the channel being prismatic). General reference for an introduction to the computation of gradually varied flow profiles is Henderson "Open Channel Flow", The MacMillan Comp., NY, (1966, Chapter 5)

Here we just present a <u>very</u> approximate procedure that can be used to obtain a rough estimate of the distance required for a gradually varied flow to achieve a given change of depth.

First, of course, we must know:

Q = Discharge
S_0 = Channel Slope
'n' $= 0.038\,\varepsilon^{1/6}$ - Manning's 'n'
$A(h)$ etc = Channel cross-section

With this information available, we can compute

$$h_n = \text{normal depth, from } S_o = S_{fn} = \frac{n^2 Q^2}{A_n^{10/3} / P_n^{4/3}}$$

and

$$h_c = \text{critical depth, from } Fr_c^2 = \frac{Q^2 b_{sc}}{g A_c^3} = 1$$

This will tell us if channel has a mild or a steep slope and give us an idea about the type of gradually varied flow profile to expect. (M-types or S-types).

Now, we assume that two depths are prescribed, say

$$h_1 = h \text{ @ } x = x_1 \;; \quad h_2 = h \text{ @ } x = x_2$$

where x_2 is unknown, i.e. we seek the distance from x_1 where the depth reaches h_2.

We have

$$\frac{h_2 - h_1}{x_2 - x_1} \approx \overline{\left(\frac{dh}{dx}\right)}_{1-2} = \text{average surface slope}$$

Now, we may obtain an estimate of the surface slope between h_1 and h_2 from the gradually varied flow equation

$$\overline{\left(\frac{dh}{dx}\right)}_{1-2} \simeq \frac{\overline{S_o} - \overline{S_f}}{1 - \overline{Fr^2}}$$

where

$$\bar{S}_f = (S_{f1} + S_{f2})/2 = n^2Q^2\left(\frac{1}{A_1^{10/3}/P_1^{4/3}} + \frac{1}{A_2^{10/3}/P_2^{4/3}}\right)/2$$

and

$$\overline{Fr}^2 = \frac{Q^2}{g}\left(\frac{b_{s1}}{A_1^3} + \frac{b_{s2}}{A_2^3}\right)/2 = (Fr_1^2 + Fr_2^2)/2$$

[an alternative would be to take the average depth $\bar{h} = (h_1 + h_2)/2$ and compute from this the corresponding $S_f = \bar{S}_f$ and $Fr^2 = \overline{Fr}^2$]

Thus, we have a rough estimate for

$$X_2 = X_1 + \frac{h_2 - h_1}{\overline{(dh/\partial x)}_{1-2}} = X_1 + \frac{(1 - \overline{Fr}^2)}{(S_0 - \bar{S}_f)}(h_2 - h_1)$$

If $h_2 - h_1$ is very large, and therefore the above approximation of a single step, very rough, one may subdivide $h_2 - h_1$ into several steps and sum the individual step length to get the desired result [this would amount to a numerical solution procedure]. Although very rough, the estimate of $X_2 - X_1$ from a single step at least will tell you if the distance required to reach h_2 is measure in 10's or 100's of meters or in km.

Before performing the actual calculations check the type of Profile expected to connect h_1 and h_2, in particular, be on the lookout for hyd. jumps along the way.