MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Civil and Environmental Engineering

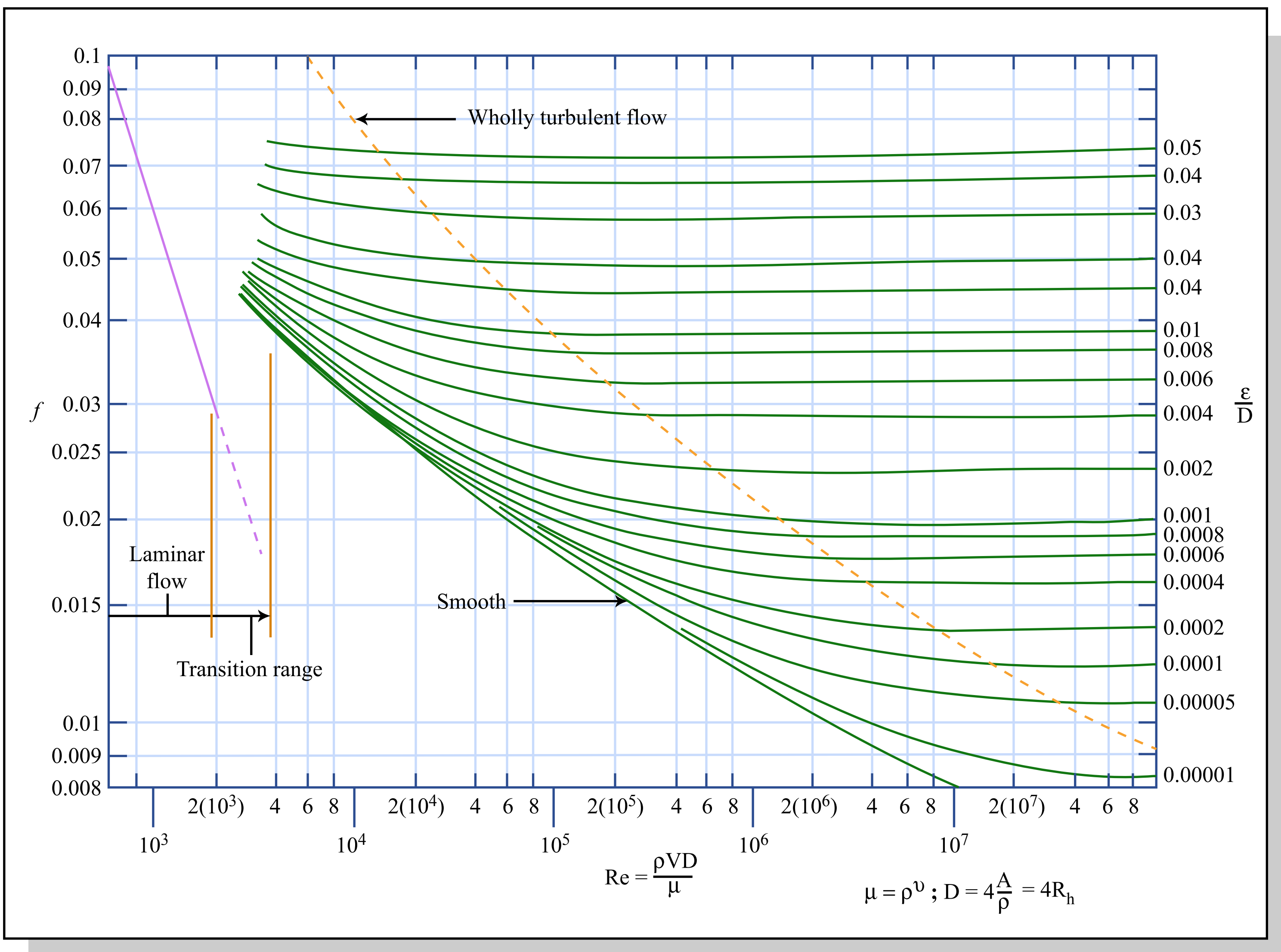
1.060/1.995 Fluid Mechanics

Tuesday, May 18, 2004
9am - 12 noon

Prof. Ole S. Madsen

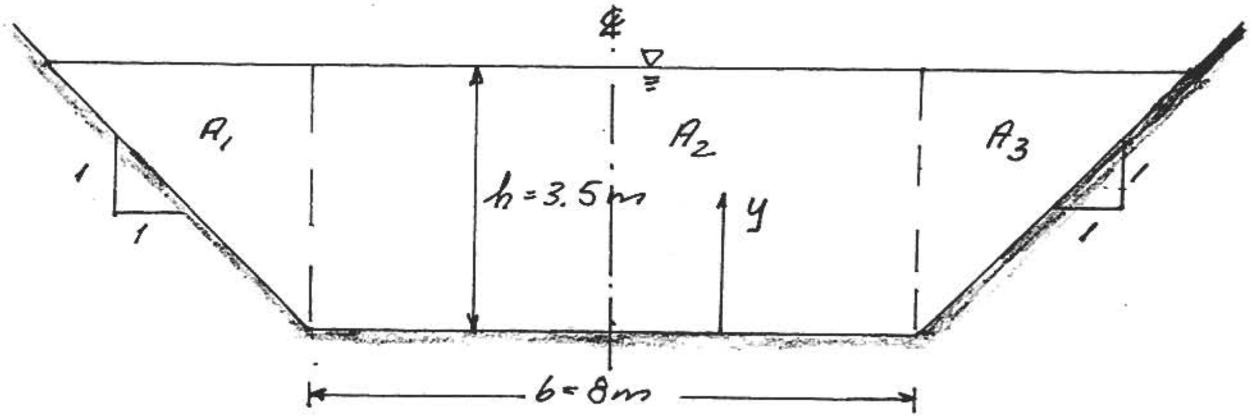
NOTE

1. There are six problems of equal weight. Be sure to allocate an appropriate amount of time for each.
2. Solutions should be expressed in terms of the problem notation and then the numerical results should be obtained.
3. Please indicate clearly, using sketches when necessary, the assumptions and definitions you are introducing in carrying out your analyses. Do not hesitate to make reasonable assumptions, but state the reason why you make them.
4. Please be as neat as possible and clearly indicate what and where your answer is (only one answer!).
5. Unless noted otherwise, the fluid is water: ($\rho = 1{,}000\ kg/m^3, \upsilon = 10^{-6}\ m^2/s$)
6. Cheat sheets #1, 2, and 3 are provided along with the Moody Diagram below.



Graph by MIT OCW.

Problem No. 1



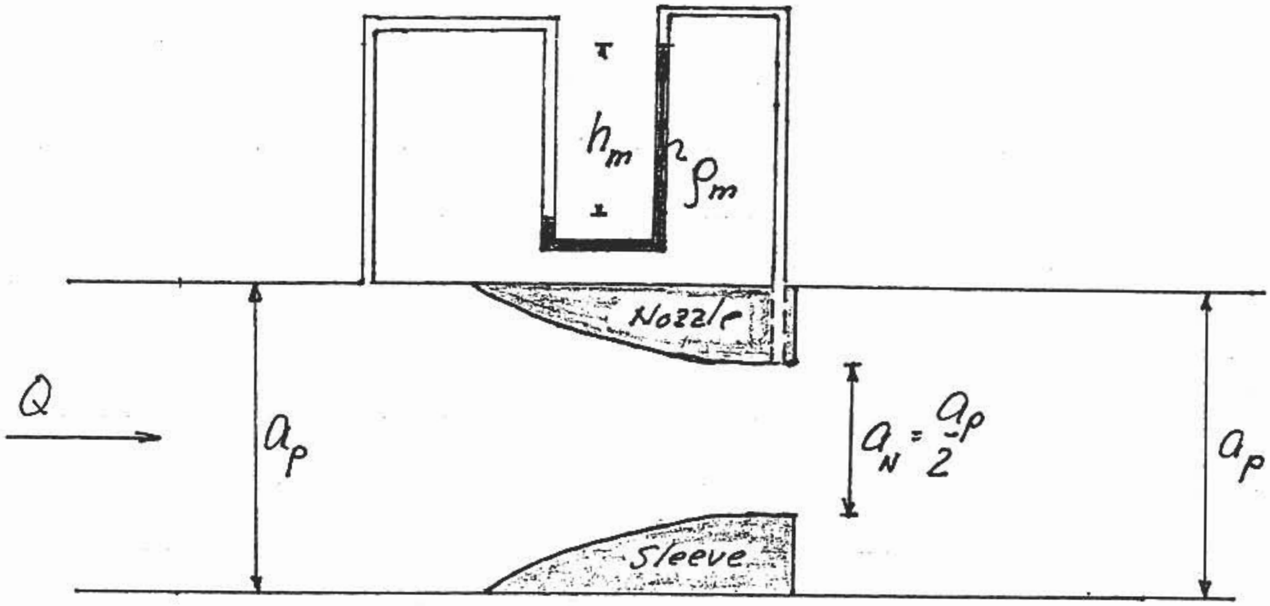
A gate is built across a trapezoidal channel of cross-section shown in sketch. The gate holds back water that rises to an elevation of h = 3.5m above the 8-m-wide horizontal bottom of the channel.

a) Determine the total pressure force, P_H, acting on the upstream, vertical face of the gate.

b) Determine the location of the center of gravity, in particular its elevation, y_{CG}, above the 8-m-wide horizontal bottom, of the trapezoidal face of the gate corresponding to h = 3.5m.

c) Determine the center of pressure, i.e. the line of action, for the total pressure force obtained in (a). [Note: Center of Pressure is not the same as Center of Gravity, since pressure varies linearly with depth, e.g. CG of a vertical rectangular area of height h is h/2 above bottom, whereas the CP is h/3 above bottom.]

Problem No. 2



The sketch shows a section of pipe in which a nozzle-sleeve is inserted in the pipe. The pipe has a square cross-section with a sidelength of $a_p = 10$ cm, and the nozzle-sleeve reduces this sidelength to $a_N = a_p/2$. A mercury manometer ($\rho_m = 13.6\rho$) is connected to the pipe as shown and gives a reading of $h_m = 6.1$ cm.

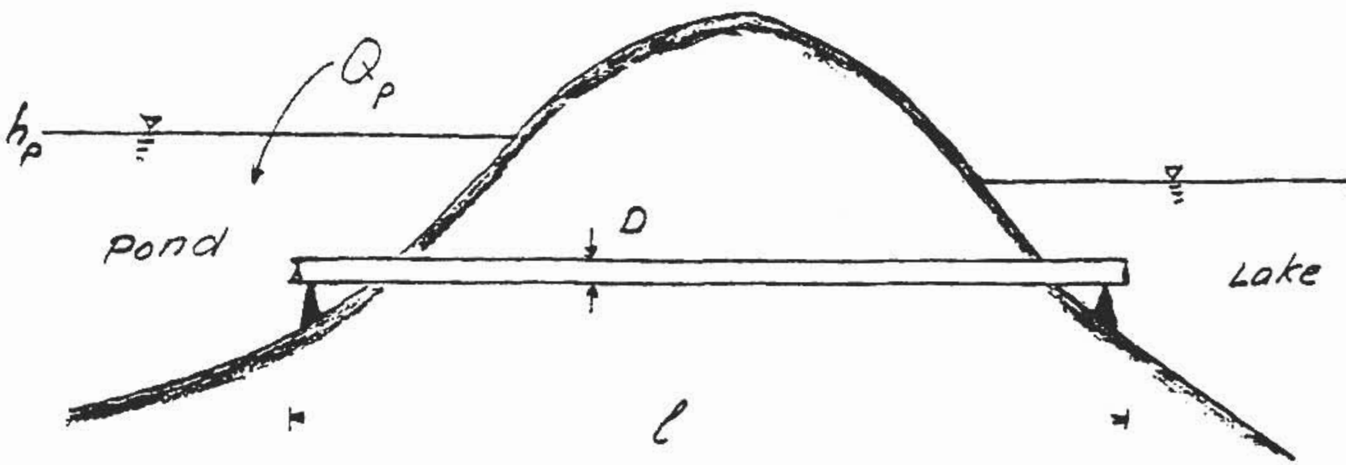
a) Determine the discharge, Q, in the pipe. [Default value $Q = 0.011$ m^3/s]

b) Determine the headloss associated with the nozzle-sleeve inserted in the pipe when wall-friction is neglected.

c) Determine the force exerted by the flow on the nozzle-sleeve (when wall-friction is neglected).

d) Assuming the pipe material to have a roughness $\varepsilon = 0.05$ mm, determine the length of pipe required to give a frictional headloss equal to the headloss computed in (b), and comment on why this length justifies the neglect of wall friction in (c).

Problem No. 3



The sketch (not to scale) shows a pond that is connected to a lake through a horizontal $\ell = 400$ m long straight circular concrete pipe, $\varepsilon = 1$ mm. The pipe inlet and outlet are located some distance above bottom, to prevent erosion. The pond receives an inflow of water, $Q_p = 0.25$ m^3/s, and it is desired to maintain the pond at a level, h_P of 1.0 m above the lake level, h_L, by proper choice of the diameter D of the concrete pipe.

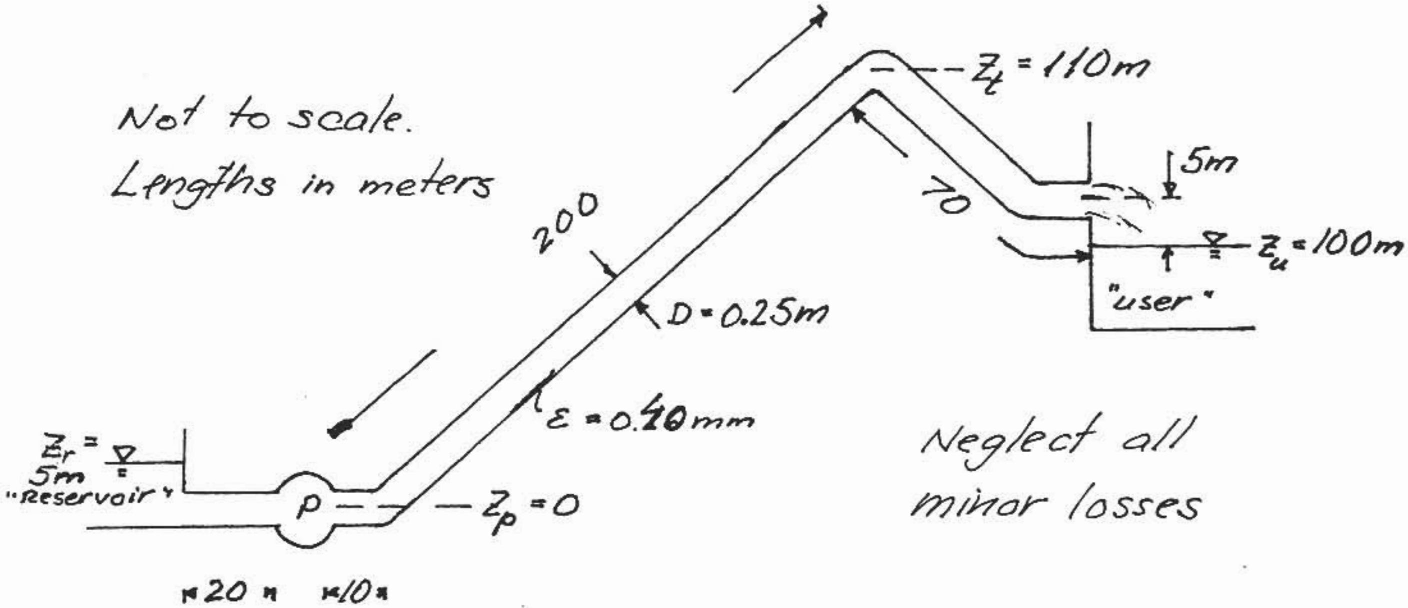
a) Set up a relationship (in general terms, i.e. using D for diameter, f for friction factor, K_L for minor loss coeffictents, etc.) between the velocity V in the pipe and the difference in water levels, $h_P - h_L$.

b) Determine the pipe diameter D which would give the required flow, Q, from pond to lake for $h_P - h_L =$ 1 m.

c) Corresponding to your solution in (b) sketch the Energy and Hydraulic Grade Lines from inlet to exit of the pipe. Although you are asked for a sketch, try to make it to scale in the vertical.

d) A graduate of Haavad College suggests that a smaller diameter would be required if the pipe was installed with a slope from the pond towards the lake instead of being horizontal. Is s/he correct?

Problem No. 4



A pump (at elevation $z_p = 0$) delivers a flow rate of $Q = 0.1\ m^3/s$ from a reservoir ($z_r = 5$ m) to a water tank (the "user") through a 25-cm-diameter cast iron pipe ($\varepsilon = 0.4$ mm). The water level in the tank is at $z_u = 100$ m with the pipe outlet 5 m above the free surface. The total length of pipe connecting reservoir and user is 300 m and <u>minor losses may be neglected</u>. The pipe passes over the top of a hill ($z_t = 110$ m) at a distance of 70 m from the user.

a) Determine the velocity in the pipe.

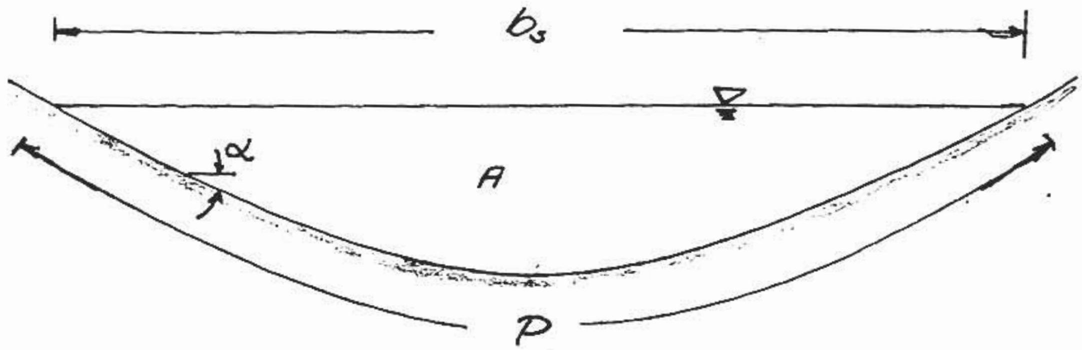
b) Determine the Darcy-Weisbach friction factor, f.

c) Determine the power (1 HP = 745 watt) required by the pump motor if the pump's efficiency is assumed to be $\eta = 0.8$.

d) Determine the pressure in the pipe at its highest elevation. $z_t = 110$ m.

e) Is the pressure determined in (d) of concern, i.e. is cavitation a potential problem, and what would happen if the pipe developed a leak at its highest point?

Problem No. 5



Typically rivers are much wider than they are deep. The sketch above shows a typical river crossection, for which the slope of the river banks, i.e. the lateral slope, is denoted by a and as a consequence of h<<b we have that a is small, say $a < 10°$.

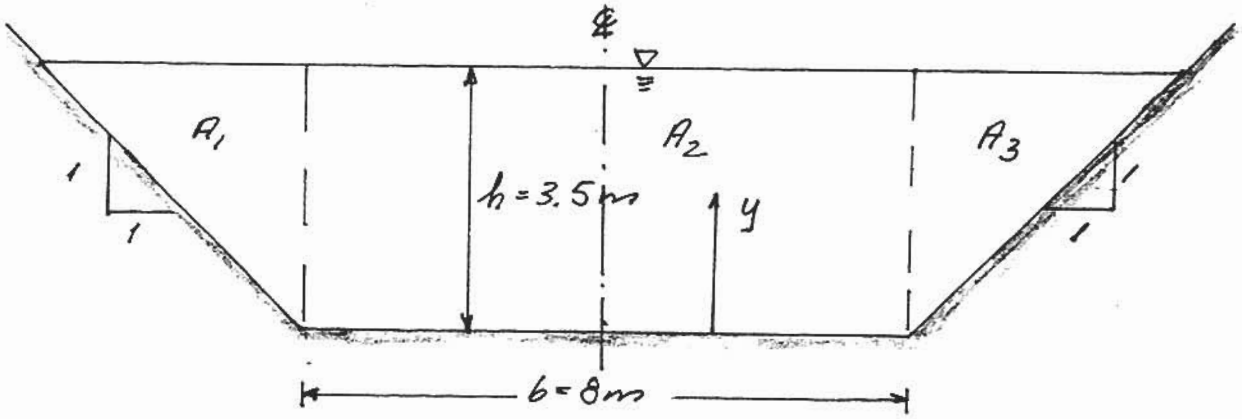
a) By balancing gravitational and frictional forces for a uniform steady flow in a prismatic channel of slope S_o, derive the classical hydraulic formula for the boundary shear stress $\tau_s = \rho g R_h S_o$

b) Show that the hydraulic radius, R_h, and the mean depth, h_m, are virtually identical for the typical river crossection discussed above.

c) Assuming $R_h = h_m$ and adopting the Darcy-Weissbach espression for the uniform steady ("normal") flow velocity in a channel of slope S_o, obtain a simple expression for the Froude Number in terms of Darcy-Weisbach's f and the slope S_o corresponding to normal flow.

d) From the result obtained in (c), obtain a rough estimate of the bottom slope, S_{oc}, for which normal flow in a "typical river" will be critical.

Problem No. 6



The sketch shows the crossection of a trapezoidal channel with a bottom width, b = 8 m, and sides sloping 1on 1. The channel roughness is estimated to be of the order $\varepsilon = 2$ cm, and its normal depth, corresponding to uniform steady flow, is $h_n = 3.5$ m. The channel is carrying a discharge of $Q = 200\ m^3/s$

a) What is the value of Manning's "n" for this channel? (Default value is "n" = 0.02 in SI-units)

b) Estimate the slope of the channel, S_0.

c) Is normal flow super or subcritical? (Justify your answer)

d) Corresponding to normal flow determine the Specific Energy [Head], E_s.

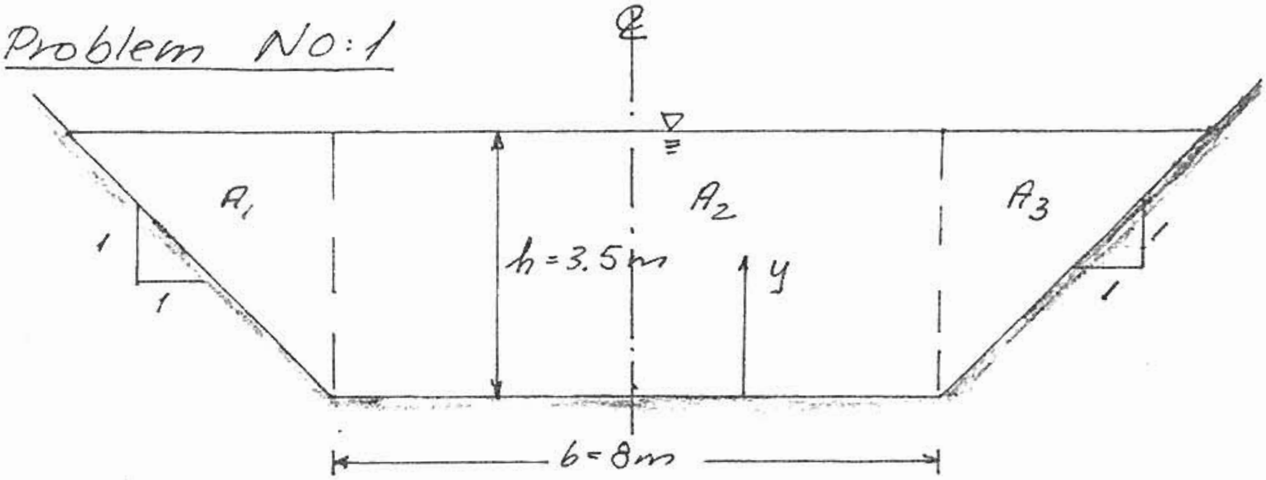
e) Corresponding to normal flow determine the Momentum and Pressure Thrust, MP. [take advantage of *the similarity between this problem and Problem No. 1*]

1.060/1.995 FLUID MECHANICS

Scheduled Final Examination, 18th May 2004

SOLUTIONS

Problem NO:1



a)

Splitting the cross-section into a retangular area (A_2) and two identical triangular areas (A_1 & A_3) we have

$A_1 = A_3 = \frac{1}{2}h^2$; $CG_1 = CG_2 = \frac{1}{3}h$ below free surface

$A_2 = b \cdot h$; $CG_2 = \frac{1}{2}h$ below free surface

$$\underline{P_H} = \sum p_{CG} A = 2 \cdot \left(\rho g \tfrac{1}{3}h\right)\tfrac{1}{2}h^2 + \left(\rho g \tfrac{1}{2}h\right)bh =$$

$$\rho g\left[\tfrac{1}{3}h^3 + \tfrac{1}{2}h^2 b\right] = 1000 \cdot 9.8\left[\tfrac{1}{3}3.5^3 + \tfrac{1}{2}3.5^2 \cdot 8\right] = \underline{620\ kN}$$

b)

The general formula for total hydrostatic pressure for reads: $P_H = \rho g (h - y_{CG}) A$, or

$$h - y_{CG} = \frac{P_H}{\rho g A} = \frac{P_H = 620\,kN}{1000 \cdot 9.8\left[2 \cdot \frac{1}{2} \cdot 3.5^2 + 3.5 \cdot 8\right]} = 1.57\,m \Rightarrow \underline{y_{CG} = 1.93\,m}$$

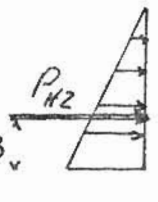
y_{CG} is measured from the 8-m-wide bottom upwards.

Note: Same result is obtained by applying the definition of Center of Gravity of an area:

$y_{CG} = \Sigma(y_{CG}A) / \Sigma(A) = [2\cdot\frac{2}{3}h\cdot\frac{1}{2}h^2 + \frac{1}{2}h\cdot hb] / [2\cdot\frac{1}{2}h^2 + hb] = 1.93\,m!$

c)

For area A_2, which is rectangular, we have the standard 2-D formula for the moment

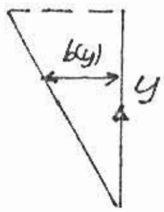


$M_2 = P_{H2}\cdot\frac{1}{3}h = \frac{1}{2}\rho g h^2 b\cdot\frac{1}{3}h = 560\ kNm$

[Note: $CG_2 = \frac{1}{2}h$ above bottom, but $CP_2 = \frac{1}{3}h$ above bottom, i.e. Center of Gravity $\neq$ Center of pressure]

For area A_1 (same as A_3), which is triangular, standard 2-D formula does not apply! We must use fundamentals.

$p(y) = \rho g(h-y)$; $b(y) = y$; arm $= y$



$$M_1 = M_3 = \int_0^h p(y)\cdot b(y)\cdot y\,dy = \rho g\int_0^h (hy^2 - y^3)\,dy =$$

$$\rho g\left[h\tfrac{1}{3}y^3 - \tfrac{1}{4}y^4\right]_0^h = \rho g\tfrac{1}{12}h^4 = 122.6\ kNm$$

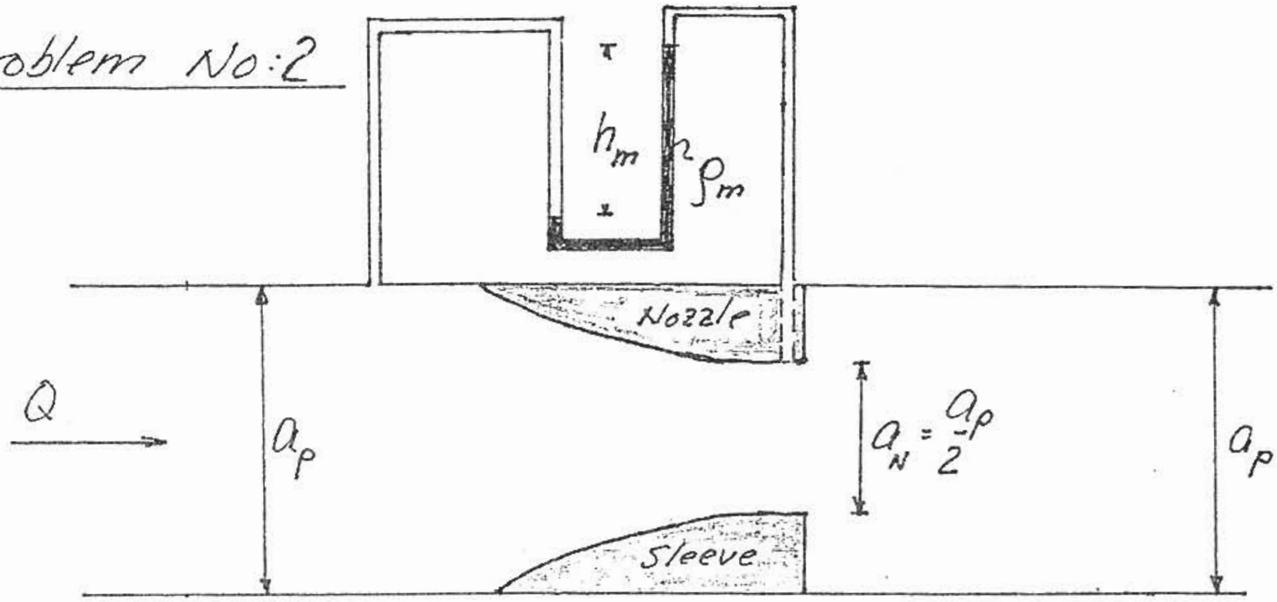
So, we have by definition of Center of Pressure

$$\underline{y_{CP}} = \frac{\Sigma M}{P_H} = \frac{560 + 2\cdot 122.6}{620} = \frac{805}{620} = \underline{1.3\,m}$$

(above bottom, located on ℄ of crossection, and acting horizontally towards the dam).

[Note: For triangular areas, the Center of Pressure is: $y_{CP1} = y_{CP2} = M_1 / (\rho g\frac{1}{3}h\,\frac{1}{2}h^2) = 122.6/70 = 1.75\,m$, i.e. CP_∇ is at mid-depth, $\frac{1}{2}h$ above bottom, whereas CG_∇ is $(2/3)h$ above bottom: $y_{CP} \neq y_{CG}$]

Problem No:2



a)

Between upstream and nozzle opening, we have a short transition of a converging flow ⇒ No headloss. Therefore, with A_p = pipe area = a_p^2 [It's a square pipe!] and $A_N = a_N^2 = (a_p/2)^2 = \frac{1}{4}a_p^2 = \frac{1}{4}A_p$, application of conservation of mass gives

$$Q = V_p A_p = V_N A_N \Rightarrow \underline{V_N = Q/A_N = 4V_p = 4\,Q/A_p}$$

and Bernoulli

$$\frac{V_p^2}{2g} + p_p/\rho g + z_p = \frac{V_N^2}{2g} + p_N/\rho g + \underset{z_p}{\overset{\shortparallel}{z_N}} + \underset{0}{\cancel{\Delta H}}$$

or

$$(p_p + \rho g z_p) - (p_N + \rho g z_N) = \frac{1}{2}\rho\left(V_N^2 - V_p^2\right) = \frac{15}{2}\rho V_p^2$$

Manometers of the type shown record difference in piezometric head, or

$$(p_p + \rho g z_p) - (p_N + \rho g z_n) = (\rho_m - \rho) g h_m$$

Introducing this in expression from Bernoulli gives

$$(\rho_m - \rho) g h_m = \frac{15}{2} \rho V_p^2 \Rightarrow V_p = \sqrt{\frac{2}{15}(\rho_m/\rho - 1) g h_m}$$

and with $\rho_m/\rho = 13.6$, and $h_m = 6\,cm = 0.061\,m$, we get

$$\underline{V_p = 1.002 \approx 1.00\,m/s} \; ; \; \underline{Q} = A_p V_p = a_p^2 V_p = \underline{0.01\ m^3/s}$$

b)

Following outflow from the nozzle opening the flow expands. This results in an expansion headloss, given by

$$\underline{\Delta H_{exp}} = \frac{(V_N - V_p)^2}{2g} = \frac{(4V_p - V_p)^2}{2g} = \frac{3^2}{2g} = \underline{0.46\,m}$$

c)

Neglecting friction, Bernoulli from upstream of nozzle-sleeve to far enough downstream to make flow welbehaved gives

$$\frac{V_p^2}{2g} + \frac{p_p}{\rho g} + z_p = \frac{V_p^2}{2g} + \frac{p_d}{\rho g} + z_d + \Delta H_{exp} + \Delta H_f^{\sim 0}$$

$$p_p - p_d = \rho g \, \Delta H_{exp}.$$

Applying the momentum principle to same control volume, we have

$$(\rho V_p^2 + p_p) A_p = \overrightarrow{MP_1} = \underbrace{(\rho V_p^2 + p_d) A_p}_{\overleftarrow{MP_2}} + \underbrace{F_s}_{\text{force from sleeve}}$$

$$\underline{F_s} = (p_p - p_d) A_p = \rho g A_p \Delta H_{exp} = \underline{45.1\ N}$$

This is the force on the nozzle-sleeve which is in direction of flow.

d)

$$A_p = a_p^2 \ ; \ P_p = \text{wetted perimeter} = 4a_p$$

$$R_h = A_p / P_p = a_p/4 \Rightarrow 4R_h = a_p \quad [\text{used for } D]$$

$$Re = \frac{4R_h V_p}{\nu} = \frac{a_p V_p}{\nu} = 10^5 ; \quad \varepsilon/4R_h = \frac{0.05}{100} = 0.0005$$

Now, from Moody:

$$f = 0.0205 \quad (\text{good eyes})$$

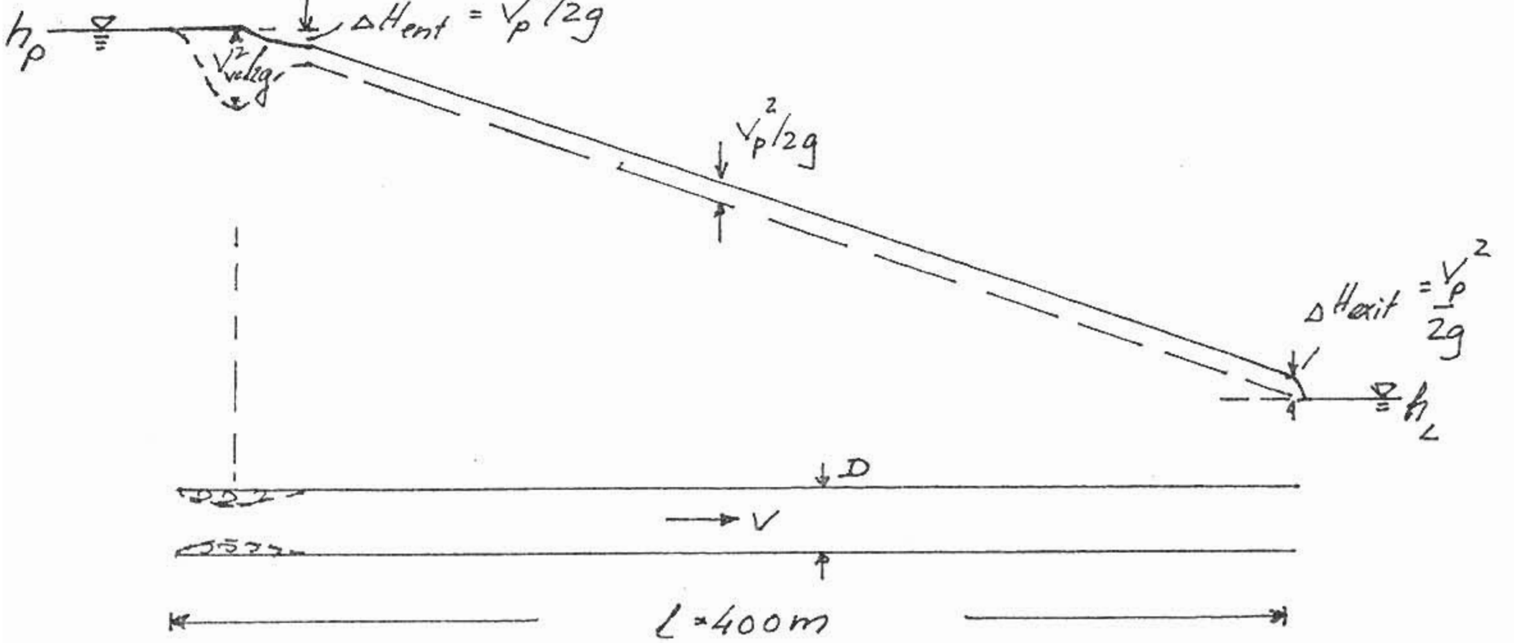
Thus,

$$\Delta H_f = f \frac{\ell_f}{\underbrace{4R_h}_{a_p}} \frac{V_p^2}{2g} = \Delta H_{exp} = 0.46 \text{ m}$$

$$\underline{\ell_f} = \Delta H_{exp} \, 2g \, a_p / (f V_p^2) = 43.98 \approx \underline{44 m}$$

44 m of pipe is required to give the same headloss as the nozzle causes. This 'enormous' length, compared to the length of the nozzle-affected flow (maybe of the order $10a_p \simeq 1 m$) justifies the negligible headloss assumed in (a) and also the neglect of shear stresses (frictional forces) in (c).

Problem No: 3



a)

Head losses between pond and lake consist of

Frictional loss $= f(l/D) V_p^2/(2g)$

Entrance loss [re-entrant inflow $C_c = 0.5$] $= K_{L,ent} V_p^2/(2g)$

Exit loss to lake $= K_{L,exit} V_p^2/2g$

$K_{L,ent} = (1/C_c - 1)^2 = 1$; $K_{L,exit} = \left(1 - \frac{A_{pipe}}{A_{Lake}}\right)^2 = 1$ ($A_{Lake} \gg A_{pipe}$)

Therefore:

$$h_p = h_L + \left(K_{L,ent} + K_{L,exit} + f\frac{l}{D}\right)\frac{V_p^2}{2g} = h_L + \left(2 + 400\frac{f}{D}\right)\frac{V_p^2}{2g}$$

$$V_p = \sqrt{2g(h_p - h_L)} \Big/ \sqrt{2 + 400f/D} \qquad (D \text{ in meters})$$

b)

$$Q = V_p A \Rightarrow V_p = Q/A = \frac{4}{\pi} Q / D^2 \Rightarrow \text{into (a) gives}$$

$$D^2 = \frac{4Q}{\pi} \sqrt{2 + 400f/D} \Big/ \sqrt{2g(h_p - h_L)}$$

or, with $Q = 0.25\, m^3/s$ and $h_p - h_L = 1m$

$$D = 0.268 \left(2 + 400f/D\right)^{1/4} \Rightarrow D^{(n+1)} = 0.268 \left(2 + 400f/D^{(n)}\right)^{1/4}$$

We must solve this by iteration, but neither "f" nor "D" is known. So, we start by taking $f = 0.02$ (our good old stand-by) and iterate to get D starting with $D^{(0)} = \infty$ (equivalent to neglect of friction)

$D^{(0)} = \infty \Rightarrow D^{(1)} = 0.32\,m \Rightarrow D^{(2)} = 0.61\,m \Rightarrow D^{(3)} = 0.53\,m \Rightarrow D^{(4)} = 0.55\,m$

So, $D \simeq 0.54\,m$ if $f = 0.02$, but is it?

$Re = D \cdot V_p / \nu = D \left[Q / \left(\frac{\pi}{4} D^2 \right) \right] / \nu = \underset{D}{0.54} \cdot \underset{V}{1.09} / 10^{-6} = 5.9 \cdot 10^5$

$\varepsilon / D = 0.001 / 0.54 = 1.9 \cdot 10^{-3}$

Moody Diagram gives: $f = 0.0225$. With this value, and starting iterations with $D^{(0)} = 0.54\,m$ we obtain:

$D^{(0)} = 0.54\,m \Rightarrow D^{(1)} = 0.56\,m \Rightarrow D^{(2)} = 0.55\,m \Rightarrow D = 0.55\,m$

We are done: Change in D from 0.54 to 0.55 m will not produce a change in Re & ε/D to change f, so: $\underline{f = 0.0225 \text{ and } D = 0.55\,m}$

c)

For $D = 0.55\,m$ we have $V_p = Q / \left(\frac{\pi}{4} D^2 \right) = 1.04 \frac{m}{s}$ ~ and therefore $\underline{V_p^2 / 2g = 0.055\,m}$. At vena contracta of inflow $V_{vc} = V_p / C_c = 2 V_p$ and $\underline{V_{vc}^2 / 2g} = 4 V_p^2 / 2g = \underline{0.22\,m}$.

$$\left.\begin{array}{l} EGL = H\,(m) \text{ above pipe } ℄ \;(z_p = 0) \\ HGL = (H - V^2/2g) \text{ above pipe } ℄ \end{array}\right\} EGL - HGL = \frac{V^2}{2g}$$

For graphical representation see sketch at start of solution to this problem.

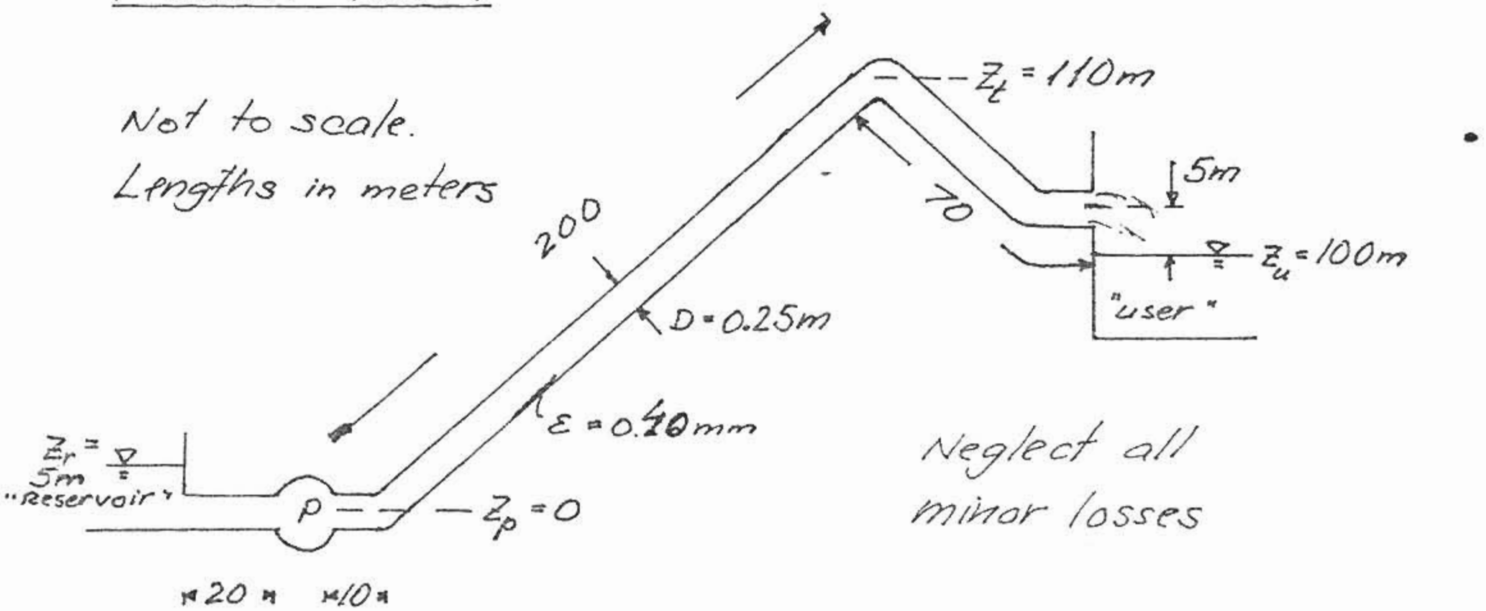
d)

So long as pipe exit is below free surface in Lake, the exit head is h_L = lake level $\Rightarrow$ actual z_{exit} does not enter problem at all $\Rightarrow$

$\underline{\text{The Haaward Student is WRONG}}$ (regardless of gender!)

Problem NO: 4

Not to scale.
Lengths in meters



Neglect all minor losses

a)

$$\underline{V} = Q/A = 0.1/\left(\tfrac{\pi}{4}0.25^2\right) = \underline{2.04\ m/s}$$

b)

$$\underline{Re} = VD/\nu = 2.04 \cdot 0.25/10^{-6} = \underline{5 \cdot 10^5};\ \underline{\varepsilon/D} = \frac{0.40 \cdot 10^{-3}}{0.25} = \underline{0.0016}$$

Moody gives : $\underline{f \cong 0.022}$

c)

Head at start $= z_r = 5$; Head at end $= (z_u + 5) + V^2/2g = 105.2m$

$\underline{H_p}$ = pump head $= 105.2 - 5 +$ Headloss $= 100.2 + f(l/D)V^2/2g =$

$$100.2 + 0.022 \cdot (300/0.25)\ 2.04^2/(2 \cdot 9.8) = \underline{105.8\ m}$$

Power supplied from pump $= \rho g Q H_p = 103.7\ kW$

Power to pump $= \underline{BHP} = \rho g Q H_p/\eta_p = 103.7/0.8 = \underline{129.6kW} = \underline{174HP}$

d)

$$H_{top} = \frac{V^2}{2g} + \underset{\text{"}110m}{z_t} + \frac{p_t}{\rho g} = (z_u + 5) + \frac{V^2}{2g} + f\,\frac{l_{t-u}}{D}\,\frac{V^2}{2g} = 105 + \left(1 + 0.022\frac{70}{0.25}\right)\frac{V^2}{2g}$$

$$p_t = \rho g\left(105 - 110 + 0.022\,\frac{70}{0.25} \cdot \frac{2.04^2}{2 \cdot 9.8}\right) = \rho g(-5 + 1.31) = -3.69\rho g$$

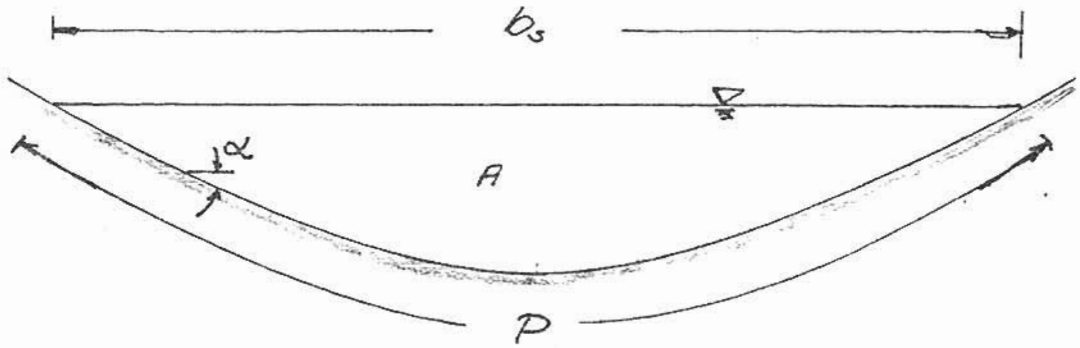
$$p_t = -36.2\ kPa \quad \text{(Gauge Pressure)}$$

e)

$p_t/\rho g = -3.69m \Rightarrow$ Not close to $-10m \Rightarrow$ <u>Cavitation not a problem</u>

If pipe has leak at z_t <u>air will be sucked</u> in since $p_t < 0$ and flow may be disrupted.

Problem No: 5



a)

Shear force parallel to bottom = $\tau_s A_s$

A_s = area on which τ_s acts = $P \Delta x \Rightarrow F_{\tau_s} = \tau_s P \Delta x$

Gravity force parallel to bottom = mass $\cdot g_x$

mass = $\rho A \Delta x$; $g_x = g \sin\beta = g S_o$ $[\underline{S_o = \sin\beta}] \Rightarrow$

$F_{gx} = \rho g A S_o \Delta x$

$F_{\tau_s} = \tau_s P \Delta x = F_{gx} = \rho g A S_o \Delta x \Rightarrow \underline{\tau_s} = \rho g (\frac{A}{P}) S_o = \underline{\rho g R_h S_o}$

b)

R_h = hydraulic radius = A/P
h_m = mean depth = A/b_s
$\Big\}$ $\underline{R_h \approx h_m \text{ if } P \approx b_s}$

line-element along free surface, δb_s, is related to line-element along bottom, δP, by $\underline{\delta b_s = \cos\alpha \, \delta P}$.

If $\alpha < 10°$ then $\cos\alpha > 0.985$, so $\delta b_s \approx \delta P$

$b_s = \sum \delta b_s \approx \sum \delta P = P \Rightarrow \underline{R_h \approx h_m}$

c)

$\tau_s = \rho g R_h S_o$ [from (a)], $\tau_s = (f/8) \rho V^2$ [D-W from Cheat Sheet]

$$\rho g R_h S_o = \frac{f}{8} \rho V^2 \Rightarrow \underline{\frac{V^2}{g R_h} \approx \frac{V^2}{g h_m} = Fr^2 = \frac{8 S_o}{f}}$$

d)

If $Fr = 1$ for normal flow, then

$Fr_n^2 = 1 = 8 S_{oc} / f \Rightarrow \underline{S_{oc} = f/8}$

With typical value for $f = 0.02$ we have

If $S_o \gtrless f/8 \sim 2.5 \cdot 10^{-3}$ Slope is $\begin{Bmatrix} \text{steep} \\ \text{mild} \end{Bmatrix}$

Problem No: 6

Channel Crossection is identical to the one discussed in Problem No: 1.

a)

Area $= A = bh + 2\frac{1}{2}h^2 = (b+h)h = 11.5 \cdot 3.5 = 40.25 m^2$

Wetted Perimeter $= P = b + 2\sqrt{2}h = 8 + 2.83 \cdot 3.5 = 17.9 m$

Hydraulic Radius $= R_h = A/P = 40.25/17.9 = 2.25 m$

Manning's "n" $= 0.038 (0.02)^{1/6} = 0.0198$ (SI-units)

$$V_n = \frac{Q}{A} = \frac{200}{40.25} = 4.97 \frac{m}{s} = \frac{1}{n} R_h^{2/3} S_o^{1/2} = \frac{2.25^{2/3}}{0.0198} S_o^{1/2}$$

$$\underline{S_o} = \left[4.97 \cdot 0.0198 / (2.25)^{2/3}\right]^2 = \underline{3.28 \cdot 10^{-3}}$$

b)

$$\text{Mean depth} = h_{mn} = \frac{A}{b_s} = \frac{A}{b+2h} = \frac{40.25}{8+2 \cdot 3.5} = 2.68 m$$

$$\underline{Fr_n} = \frac{V_n}{\sqrt{g h_{mn}}} = \frac{4.97}{9.8 \cdot 2.68} = \underline{0.97 < 1} \quad \text{(but not by a lot)}$$

[Note: If h_m is incorrectly taken as $h = 3.5 m \Rightarrow Fr_n = 0.85$]

$Fr_n < 1$ means that normal flow is subcritical

c)

$$\underline{E_{sn}} = h + \frac{V_n^2}{2g} = 3.5 + \frac{4.97^2}{2 \cdot 9.8} = 3.5 + 1.26 = \underline{4.76 m}$$

d)

From Problem No: 1 we have: $P = P_H = 620 kN$

The momentum force is: $M = \rho V_n^2 A = 1000 \cdot 4.97^2 \cdot 40.25 =$ 994 kN

$$\underline{MP} = M + P = 994 + 620 = \underline{1,614 kN}$$