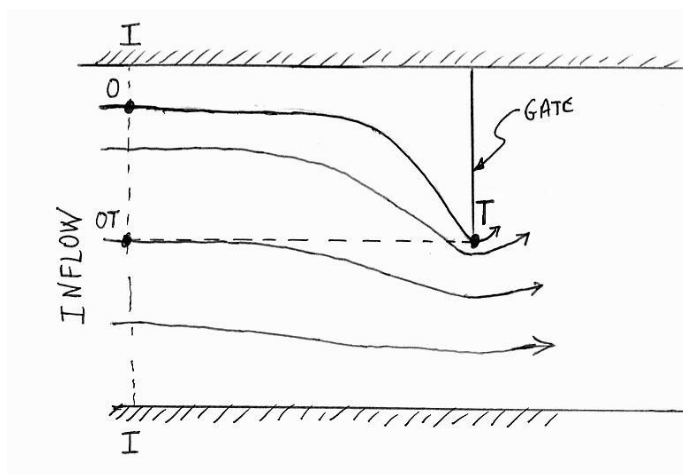
PROBLEM SET 3 - SOLUTIONS

Comments on Problem Set 3

PROBLEM 1:

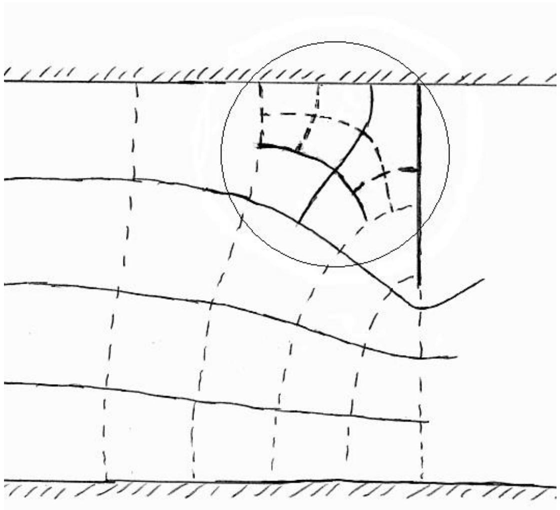
- Remember that we want the flownet to be formed by square cells. Many of your flownet cells in the region next to the gate didn't look very square. This means that your streamlines are not right (so you are not capturing how the fluid flows) and that you don't have enough precision to obtain v_t from the flownet in part **c**. Some groups got a weird-looking flownet because they forced the streamlines to be equally spaced under the gate (see second sketch in next page). This doesn't happen in reality, so these groups were unable to get square cells. In reality, the flow is only uniform far upstream and far downstream the gate. In the gate region, since all the flow is forced to pass under the gate, we will have a larger velocity next to the tip of the gate than next to the bottom of the duct. For this reason, the streamlines are not equally spaced in this region, but closer to each other next to the tip of the gate.

- In part **b**, $v_0 = v_1$ because of conservation of volume. The only condition required for conservation of volume to be true is that the fluid is incompressible. Some of you claimed that $v_0 = v_1$ only if we neglect viscosity. This is not the case: $v_0 = v_1$ is true even if we have viscous effects.

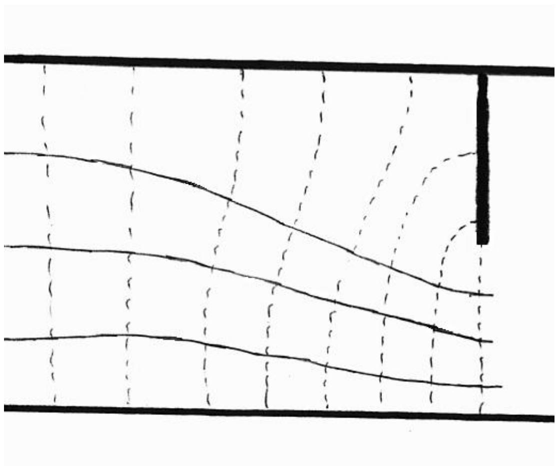


- Bernoulli equation is only applicable between two points on the same streamline!! To relate the pressure at "T" with the pressure at "0T" (see sketch) on part **d**, many groups applied Bernoulli between the point "0T" and "T". However, the flownet shows you that "0T" and "T" are on different streamlines, so you cannot apply Bernoulli to relate their pressure values! It happens that, in this problem, the result you get by applying Bernoulli between "0T" and "T" is the right one (but this is by luck!) The reason of this "lucky coincidence" is that all points at the inflow (section I-I) have the same value of $p+\rho gh$ (because flow is well behaved in I-I and pressure varies hydrostatically), and also the same value of the velocity (because flow is uniform at the inflow). However, this is just luck, and in general you can only apply Bernoulli between two points on the same streamline.

- Many people got confused about how to solve "weirdos" (the non-square looking cells next to corners). Prof. Madsen said that you can <u>check</u> whether your "weirdo" is ok by subdividing it (i.e., interpolating one streamline and one equipotential line) and seeing if the "sub-weirdos" look square. However, this is only going to work if your "weirdo" has about the same "horizontal" and "vertical" dimensions to start with! Some of you got "weirdos" that looked very far from squares (with one dimension much larger than the other) and tried to justify that they were ok by interpolating a streamline and dividing the rectangular "weirdo" into two squares… Nonononono… You have to interpolate one streamline AND one equipotential line, otherwise you are cheating! If your "weirdo" looks very non-square (i.e., long and thin) to start with, you have to modify your flownet. Let's analyze a couple of examples:



This is an example of a reasonably good flownet. I have subdivided the upper-right weirdo (the one inside the circle) twice. First, I introduced a new streamline AND a new equipotential line (both are drawn as solid lines) to get four sub-weirdos that look reasonably square. Again, I subdivided the two upper sub-weirdos (now I used dashed lines) and get sub-sub-weirdos that are again reasonably square. Therefore, I conclude that the flownet looks right.



This is an example of a bad flownet (copied from one of the answers to the problem set). The upper right weirdos look very long and thin, that is, not square at all. No matter how you try to subdivide them, there is no way you can get sub-weirdos that look square (unless you cheat interpolating streamlines only!). At the same time, the lower right cells look quite short and fat. The way of fixing this flownet is by re-drawing the streamlines upwards, and modifying the equipotential lines accordingly, to get something similar to the flownet in the previous example. Note that this correction is going to make the streamlines closer to each other near the tip of the gate, and therefore the value of v_t from the corrected flownet is going to be larger.

PROBLEM 2:

- Most groups did well on this problem. However, most of you didn't explain why the highest point of the siphon is the most critical one (i.e., has the smallest pressure). To justify this, you apply Bernoulli along the pipe (you can do this because the streamlines go along the pipe). Since $p+ \rho gh + \rho v^2/2$ is constant along the pipe, and continuity dictates that velocity is the same everywhere, the pressure will be smallest when h is highest, i.e., the smallest pressure happens at the highest point. I guess most of you knew this, but many of you didn't write it down (and lost a couple of points). As we (David and Sal) have already said a few times, many of the groups should give more details about their assumptions and calculations.

PROBLEM 3:

- You also did pretty well on this one, in general. Remember that if V(centerline) is within 3% of V(average), this means that $0.97\ V(ave) \leq V(cl) \leq 1.03\ V(ave)$. Some groups considered only one of the two limits. And again, try to give more detailed explanations of your solution procedure. Many groups didn't explain very well how they got the result from the spreadsheet (the problem said "be sure to include sample calculations"!) And it's considered not being nice with the TA to hand in a problem with a bunch of numbers and equations and not a single word…

COMMENTS ON PROBLEMS 4 TO 6:

Almost all groups made many careless math errors throughout (i.e. negative signs, etc). Always remember to go back and make sure your answer makes physical sense - this will help you catch some of these errors.

Problem #4:
For parts c and d, the change in H over change in time is equal to (-) w_i. A few groups forgot the negative sign here - it indicates a downwards velocity (assuming positive is in the upwards direction). For part f, many groups forgot the extra r when integrating over the area (2*pi*r*dr). Also, when substituting d_i/2 in for r in part f, some groups forgot to also raise the denominator to the power (i.e. r^4 = d_i^4/16).

Problem #5:
It seemed that a few groups had a bit of trouble developing the underlying concept of the problem - that the rate of change of the number of particles = (inflow + source) - (outflow + sink). Thus, the governing equation ends up as a differential equation.

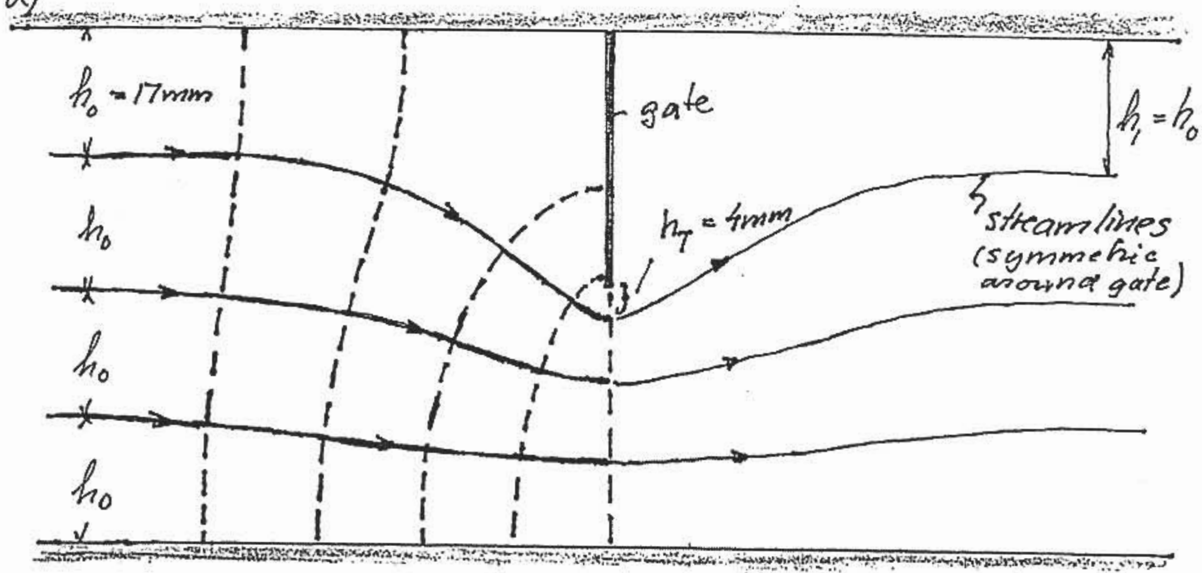
Problem #6:
Many groups got a bit tripped up with calculating the angle in part b. This was mainly because of radian -> degree conversion or because they forgot to convert omega into rad/s from rps. On the last part, some groups got an incorrect value for lift force which made it seem like the model was valid mathematically. However, no qualitative reasoning were provided to back up the calculations. Always revisit your assumptions when commenting on how well a model approximates. Even if you think your answer is correct, taking a look at the assumptions (i.e. modeling a sphere as a cylinder) might show some incosistency.

GOOD LUCK ON THE TEST!

PROBLEM SET 3 - SOLUTIONS

- PROBLEM N° 1:

a)



Notice: Flownet is symmetrical with respect to the line along the gate. Streamlines must therefore pass under the gate and be perpendicular to gate's continuation (if they were not, the streamlines would exhibit a sharp corner — a kink — directly below the gate).

b)

$v_0 = v_1$ since the streamlines far from the gate divide the duct into 4 equal streamtubes spaced $h_0 = h_1 = \frac{h}{4}$ apart, each carrying $\Delta Q = Q/4$. Then:

$$v_0 = \frac{\Delta Q}{h_0} = \frac{\Delta Q}{h_1} = v_1$$

Even easier, $v_0 = v_1$ due to the symmetry of the flow net.

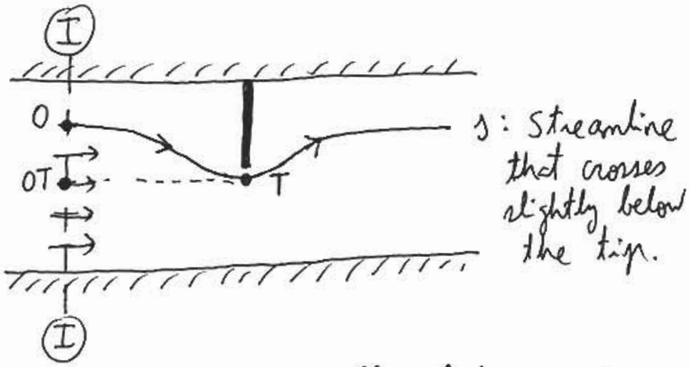
c)

From the flow net we get $h_0 = h_1 = 17$ mm and the distance from the tip of the gate to the first streamline crossing below the tip is $h_T = 4$ mm. Thus, since $v_i = \dfrac{\Delta Q}{h_i}$ and ΔQ is constant along a streamtube, we have:

$$v_0 = \frac{\Delta Q}{h_0}, \quad v_t = \frac{\Delta Q}{h_T} \Rightarrow \underline{\underline{v_t}} = \frac{h_0}{h_T} v_0 = \frac{17}{4} v_0 = \underline{\underline{4'25\, v_0}}$$

<u>Note</u>: v_T is the velocity <u>slightly below the tip</u>, in the approximation afforded by a flow net with 4 stream tubes. If you were to take more and more streamtubes and consider a point closer and closer to the tip, you would get $v_t \to \infty$.

d)



We apply Bernoulli between O and T, which are located along the same streamline "s", to obtain

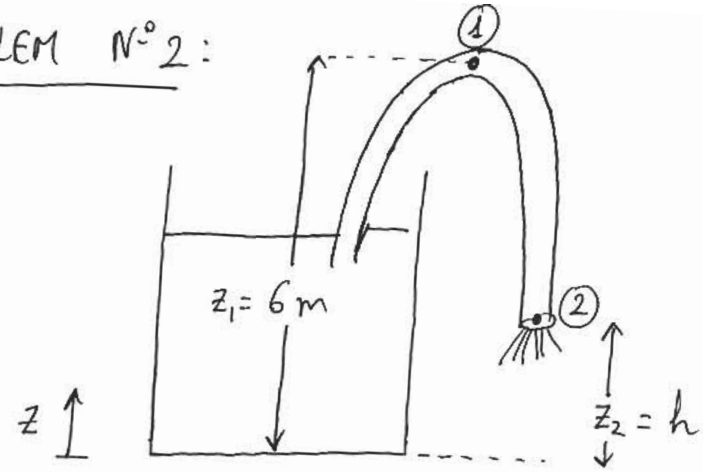
$$\frac{1}{2}\rho v_0^2 + p_0 + \rho g z_0 = \frac{1}{2}\rho v_t^2 + p_T + \rho g z_T$$

Since the flow at the inflow, I-I, is well behaved, we have hydrostatic pressure. Thus: $p_0 + \rho g z_0 = p_{OT} + \rho g z_T$.

Then:
$$\frac{1}{2}\rho v_0^2 + p_{OT} + \cancel{\rho g z_T} = \frac{1}{2}\rho v_t^2 + p_T + \cancel{\rho g z_T}$$

$$\underline{\underline{p_T}} = p_{OT} + \frac{1}{2}\rho v_0^2 - \frac{1}{2}\rho (4'25\, v_0)^2 = \underline{\underline{p_{OT} - 8'53\,\rho v_0^2}}$$

-PROBLEM N° 2:



Since all points within the tube are on the same streamline, Bernoulli implies

$$\frac{1}{2}\rho v^2 + p + \rho g z = \text{constant}$$

Furthermore, continuity in the tube implies $v = \text{constant}$.
Thus,

$$p + \rho g z = \text{constant along the tube} \quad (\star)$$

That is, the lowest pressure occurs at the point of maximum z, 1. Applying $\star$ between 1 and 2:

$$p_1 + \rho g z_1 = \underbrace{p_2}_{0 \text{ (atmospheric)}} + \rho g h$$

$$p_1 = \rho g (h - z_1) \geq -25000 \text{ Pa}$$

$$\underline{\underline{h}} \geq 6 - \frac{25000}{1000 \cdot 9'8} = \underline{\underline{3'45\ m}}$$ is the minimum value of h allowed.

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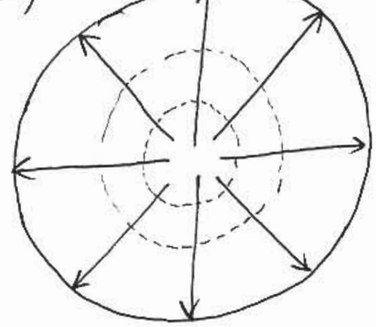
Please see:
Problem 3.22 in Munson, Bruce R., Donald F. Young, and Theodore H. Okiishi. *Fundamentals of Fluid Mechanics*. Instructor's Manual. 2nd ed. New York, NY: John Wiley & Sons, Inc., 1994, pp. 3-20.

- PROBLEM N° 4:

a)



For reasons of angular symmetry, the flow must be purely in the radial direction, i.e., streamlines are straight lines in the radial direction.

b)

Initially the outer cylinder is filled up to level h_0 $\Rightarrow$

$$\Rightarrow \text{Initial volume} = \frac{\pi}{4} d_0^2 h_0$$

Later, this same fluid is contained in the space between the two cylinders $\Rightarrow$

$$\Rightarrow \text{Final volume} = \underbrace{\frac{\pi}{4} d_i^2 H}_{\text{volume under smaller cylinder}} + \underbrace{\frac{\pi}{4}(d_0^2 - d_i^2) h}_{\text{volume in annular space between cylinders}}$$

Since final volume is equal to initial volume,

$$\frac{\pi}{4} d_0^2 h_0 = \frac{\pi}{4} d_i^2 H + \frac{\pi}{4}(d_0^2 - d_i^2) h \Rightarrow$$

$$\Rightarrow h = h_0 + \frac{d_i^2}{d_0^2 - d_i^2}(h_0 - H) = \frac{h_0 d_0^2 - H d_i^2}{d_0^2 - d_i^2} = H + \frac{d_0^2 (h_0 - H)}{d_0^2 - d_i^2}$$

(equivalent expressions).

c)

Volume under the inner cylinder $= \forall_i = \frac{\pi}{4} d_i^2 H$

As the inner cylinder is pushed down, V_i decreases:

$$\frac{\partial \forall_i}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\pi}{4} d_i^2 H\right) = \frac{\pi}{4} d_i^2 \frac{\partial H}{\partial t} = -\frac{\pi}{4} d_i^2 W_i$$

where W_i = downward velocity of inner cylinder.

Since no fluid is being added, the decrease in $\forall_i$ is equal to the volume of fluid leaving from under the inner cylinder, i.e.,

$$\frac{\partial \forall_i}{\partial t} = -V_i \cdot A_{out} \quad \text{where } A_{out} = \underbrace{\pi d_i}_{\text{circumference}} \cdot \underset{\uparrow \text{height}}{H}$$

Equating the two expressions for $\frac{\partial \forall_i}{\partial t}$,

$$V_i \pi d_i H = \frac{\pi}{4} d_i^2 W_i \Rightarrow \underline{\underline{V_i = \frac{d_i}{4H} W_i}}$$

d)

Consider a cylindrical volume of fluid under the inner cylinder, of radius $r < \frac{d_i}{2}$ and height H.

The volume of this cylinder is $\forall(r) = \pi r^2 H$.

The area (of this cylindrical volume) where outflow occurs is $A_{out}(r) = 2\pi r H$

Rate of increase in volume = $\cancel{\text{inflow}}^{\,0}$ − outflow

$$\frac{\partial}{\partial t}\left[\forall(r)\right] = -A_{out}(r)\cdot V(r)$$

$$\frac{\partial}{\partial t}\left[\pi r^2 H\right] = -2\pi r\, H\, V(r)$$

$$\pi r^2 \frac{\partial H}{\partial t} = -2\pi r\, H\, V(r) \quad ; \quad \frac{\partial H}{\partial t} = -W_i$$

$$\underline{\underline{V(r) = \frac{r}{2H} W_i}}$$

i.e., $\underline{\underline{V(r) \text{ varies linearly with } r}}$

e) Applying Bernoulli's equation along radial streamline along the bottom of inner cylinder (assuming steady flow):

$$\frac{1}{2}\rho V(r)^2 + p(r) + \rho g H = \frac{1}{2}\rho V_i^2 + p_i + \rho g H$$

($z=0$ at the bottom of the outer cylinder)

Given $p_i = \rho g(h-H)$,

$$\frac{1}{2}\rho V(r)^2 + p(r) + \rho g H = \frac{1}{2}\rho V_i^2 + \rho g h \Rightarrow$$

$$\Rightarrow \underline{\underline{p(r) = \rho g(h-H) + \frac{1}{2}\rho\left(V_i^2 - V(r)^2\right)}}$$

f)

Neglecting the inner cylinder's weight, the force with which the cylinder is being pushed downwards has to counteract the pressure force,

$$\underline{\underline{F}} = \int_0^{d_i/2} p(r)\, 2\pi r\, dr =$$

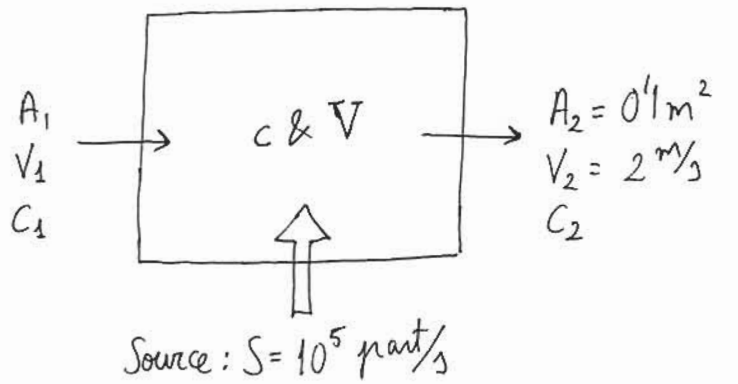
$$= \int_0^{d_i/2} \left[\rho g (h-H) + \frac{1}{2} \rho \left(V_i^2 - V(r)^2 \right) \right] 2\pi r\, dr =$$

$$= 2\pi \int_0^{d_i/2} \left[\rho g (h-H) r + \frac{1}{8} \rho \frac{W_i^2}{H^2} \left(\frac{d_i^2}{4} r - r^3 \right) \right] dr =$$

$$= 2\pi \rho g (h-H) \frac{d_i^2}{8} + \frac{\pi}{4} \rho \frac{W_i^2}{H^2} \left(\frac{d_i^4}{32} - \frac{d_i^4}{64} \right) =$$

$$= \underbrace{\rho g \frac{\pi d_i^2}{4} (h-H)}_{\text{buoyancy force}} + \underbrace{\frac{\pi}{256} \rho \frac{W_i^2 d_i^4}{H^2}}_{\text{extra force necessary to squeeze out the fluid due to the downward velocity of inner cylinder.}}$$

- PROBLEM Nº 5



V = volume of the room $= 6 \cdot 8 \cdot 2'5 = 120\ m^3$

C_1 = concentration of particles in the inflow $= 0$

C = " " " inside the room

C_2 = " " " in the outflow $= C$ (We assume the air in the room to be well mixed)

a) Conservation of the # of particles:

RATE OF CHANGE OF # OF PARTICLES INSIDE THE ROOM = (INFLOW + SOURCE) − (OUTFLOW + SINK)

$$\frac{d(cV)}{dt} = (\cancel{C_1 A_1 V_1}^{\,0} + S) - (C_2 A_2 V_2 + 0)$$

(V is constant) $(C_1 = 0)$ $(C_2 = C)$

$$V \frac{dc}{dt} = S - c A_2 V_2$$

$$\boxed{V \frac{dc}{dt} + A_2 V_2 c = S}$$

or, plugging in the numerical values,

$$\boxed{120 \frac{dc}{dt} + 0'2 c = 10^5}$$

C in part/m^3
t in seconds

b) Steady-state: $\frac{d}{dt} = 0 \Rightarrow$ The ODE reads $A_2 V_2 c = S \Rightarrow$

$$\Rightarrow \underline{\underline{V_2}} = \frac{S}{A_2 c} = \frac{10^5}{0'1 \cdot 275000} = \underline{\underline{3'64 \text{ m/s}}}$$

c) We need to solve the inhomogeneous ODE

$$V \frac{dc}{dt} + A_2 V_2 c = S$$

subjected to the initial condition $c=0$ at $t=0$. The most general solution of the ODE is

$$c = c_h + c_p$$

where c_p is a particular solution, e.g., $c_p = \frac{S}{A_2 V_2}$, and c_h is the solution of the homogeneous equation

$$V \frac{dc_h}{dt} + A_2 V_2 c_h = 0$$

$$\frac{dc_h}{c_h} = -\frac{A_2 V_2}{V} dt \Rightarrow \ln c_h = -\frac{A_2 V_2}{V} t + \underset{\substack{\uparrow \\ \text{constant}}}{K_0} \Rightarrow c_h = \underset{\substack{\uparrow \\ \text{constant}}}{K} e^{-\frac{A_2 V_2}{V} t}$$

Therefore,

$$c = K e^{-\frac{A_2 V_2}{V} t} + \frac{S}{A_2 V_2}$$

Applying the initial condition,

$$c(t=0) = K + \frac{S}{A_2 V_2} \Rightarrow K = -\frac{S}{A_2 V_2} \Rightarrow c = \frac{S}{A_2 V_2} \left(1 - e^{-\frac{A_2 V_2}{V} t}\right)$$

<u>NOTE</u>: $\lim_{t \to \infty} c = \frac{S}{A_2 V_2}$, as obtained in (b).

Now:

$$c = 0'99 \, c_{t \to \infty} \Rightarrow 1 - e^{-\frac{A_2 V_2}{V} t} = 0'99$$

$$e^{-\frac{A_2 V_2}{V} t} = 0'01$$

$$-\frac{A_2 V_2}{V} t = \ln 10^{-2}$$

$$\underline{\underline{t}} = \frac{2V}{A_2 V_2} \ln 10 = \frac{2 \cdot 120}{0'1 \cdot 3'64} \ln 10 \approx 1518 \text{ s} \approx \underline{\underline{25 \text{ min}}}$$

- PROBLEM N° 6:

a)

$$\underline{\underline{u_r}} = \frac{\partial \phi}{\partial r} = \underline{\underline{-U\left(1-\frac{a^2}{r^2}\right)\cos\theta}}$$

$$\underline{\underline{u_\theta}} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \underline{\underline{U\left(1+\frac{a^2}{r^2}\right)\sin\theta + \frac{a^2\Omega}{r}}}$$

b) At a stagnation point, $u_r = 0$ and $u_\theta = 0$ simultaneously.

$$u_r = 0 \Rightarrow \begin{cases} \cos\theta = 0 \Rightarrow \begin{cases} \theta = \frac{\pi}{2} & (1) \\ \text{or } \theta = \frac{3\pi}{2} & (2) \end{cases} \\ \text{or} \\ 1-\frac{a^2}{r^2} = 0 \Rightarrow \quad r = a \quad (3) \end{cases}$$

Case (1): $\theta = \frac{\pi}{2}$

$$u_\theta = U\left(1+\frac{a^2}{r^2}\right) + \frac{a^2\Omega}{r} > 0 \quad \forall r \Rightarrow \text{No stagnation point.}$$

Case (2): $\theta = \frac{3\pi}{2}$

$$u_\theta = -U\left(1+\frac{a^2}{r^2}\right) + \frac{a^2\Omega}{r} = 0 \Rightarrow Ur^2 - a^2\Omega r + Ua^2 = 0$$

$$r = \frac{a^2\Omega \pm \sqrt{a^4\Omega^2 - 4U^2a^2}}{2U}$$

NOTE: $\Omega = 25\,rps = 157'08\,\frac{rad}{s}$

Discriminant $= a^4\Omega^2 - 4U^2a^2 =$

$= (3'75\cdot 10^{-2})^4 \cdot 157'08^2 - 4\cdot 30^2\cdot (3'75\cdot 10^{-2})^2 =$

$= -5'01 < 0 \Rightarrow$ No real solutions $\Rightarrow$

$\Rightarrow$ No stagnation point.

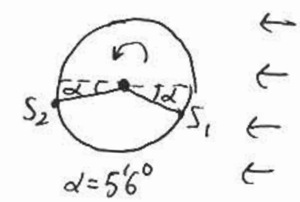
Case (3): $r = a$

$$u_\theta = 2U\sin\theta + a\Omega = 0 \Rightarrow \sin\theta = -\frac{a\Omega}{2U} = -\frac{0'0375 \cdot 157'08}{2 \cdot 30} = -0'0982 \Rightarrow$$

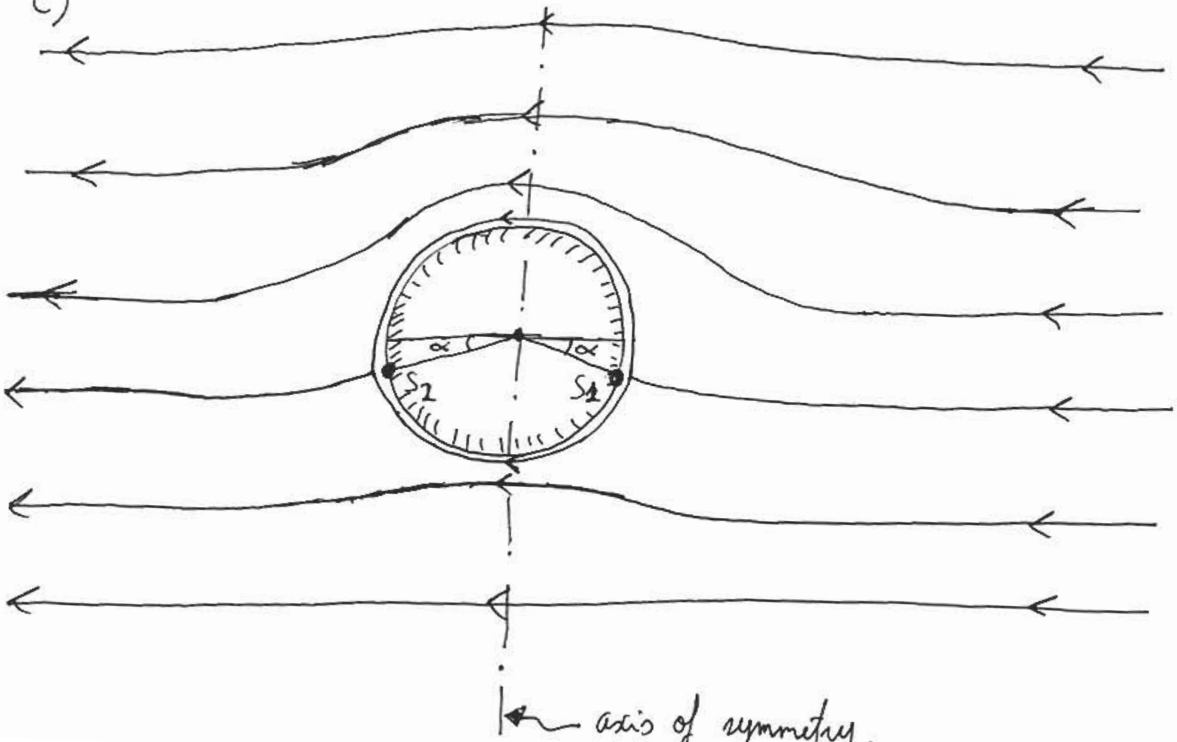
$$\Rightarrow \theta = \begin{cases} -5'6^\circ \\ 185'6^\circ \end{cases}$$

In summary,

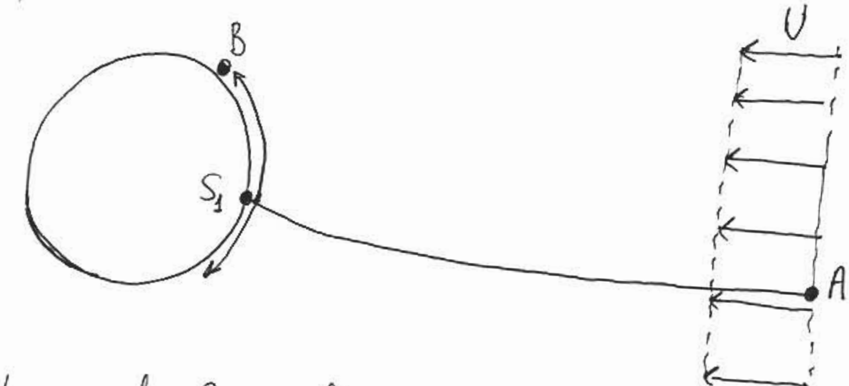
STAGNATION POINTS $\begin{cases} \underline{\underline{r = a \text{ and } \theta = -5'6^\circ}}\ (S_1) \\ \underline{\underline{r = a \text{ and } \theta = 185'6^\circ}}\ (S_2) \end{cases}$



c)



d)



We apply Bernoulli between A and B:

$$\cancelto{0}{p_A} + \rho g z_A + \frac{1}{2}\rho v_A^2 = p_B + \rho g z_B + \frac{1}{2}\rho v_B^2$$

Since gravity is neglected in this problem, $\rho g z_A = \rho g z_B = 0$, and

$$p_B = \frac{1}{2}\rho\left(v_A^2 - v_B^2\right) = \frac{1}{2}\rho\left(U^2 - v_B^2\right)$$

$$v_B^2 = u_{r_B}^2 + u_{\theta_B}^2 \underset{\text{at } B,\ r=a}{=} 0^2 + (2U\sin\theta + a\Omega)^2 =$$

$$= 4U^2\sin^2\theta + (a\Omega)^2 + 4a\Omega U\sin\theta$$

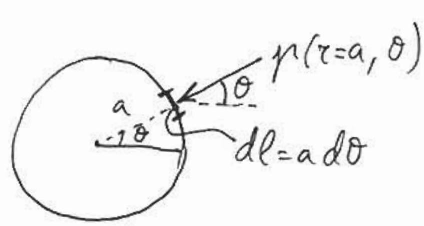
Therefore,

$$\underline{\underline{p_B}} = \frac{1}{2}\rho\left[U^2 - \left(4U^2\sin^2\theta + (a\Omega)^2 + 4a\Omega U\sin\theta\right)\right] =$$

$$= \underline{\underline{\frac{1}{2}\rho\left[U^2(1 - 4\sin^2\theta) - (a\Omega)^2 - 4a\Omega U\sin\theta\right]}}$$

e)

$$L = \int_0^{2\pi} -p\sin\theta \cdot \underbrace{a\,d\theta}_{dl} =$$



$$= -\int_0^{2\pi} \frac{1}{2}\rho\left[U^2(1 - 4\sin^2\theta) - (a\Omega)^2 - 4a\Omega U\sin\theta\right]\sin\theta\, a\, d\theta =$$

$$= -\frac{\rho a}{2}\left\{\left[U^2 - (a\Omega)^2\right]\int_0^{2\pi}\sin\theta\, d\theta - 4a\Omega U\int_0^{2\pi}\sin^2\theta\, d\theta - 4U^2\int_0^{2\pi}\sin^3\theta\, d\theta\right\}$$

Let's evaluate the trigonometric integrals:

$$\int_0^{2\pi} \sin\theta \, d\theta = -\left[\cos\theta\right]_0^{2\pi} = -(1-1) = 0$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta = \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \pi$$

$$\int_0^{2\pi} \sin^3\theta \, d\theta = \int_0^{2\pi} \sin\theta\,(1-\cos^2\theta)\, d\theta = \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{2\pi} = 0$$

Therefore,

$$\underline{\underline{L}} = \frac{\rho a}{2} 4 a \Omega U \pi = \underline{\underline{\rho U 2\pi \Omega a^2}}$$ lift force per unit length.

f)

$$\underline{\underline{F_{lift}}} = L \cdot \ell = (1'2 \cdot 30 \cdot 2\pi \cdot 157'08 \cdot 0'0375^2) \cdot (2 \cdot 0'0375) = \underline{\underline{3'75 \text{ N}}} \quad !!$$

$(\rho_{air} \approx 1'2 \, kg/m^3)$

$$\frac{F_{lift}}{Weight} \approx 2'7 \gg 0'33 \rightarrow$$ Our model, gives a physical explanation of the lift force on a baseball, but overpredicts its value by about one order of magnitude. The model is too simplistic (sphere approximated by a cylinder, smooth surface, no turbulence... all these are unrealistic assumptions).