1.054/1.541 Mechanics and Design of Concrete Structures (3-0-9)

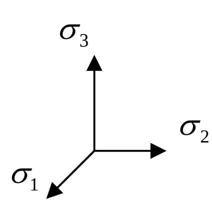
Outline 3

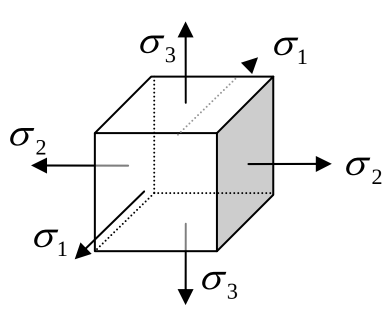
Failure Theories and Concrete Plasticity

- Failure of concrete
  - Concrete is a brittle material which fails through brittle cleavage (splitting) at the interfaces and in mortar except for high triaxial compression where shear slippage occurs resulting in a ductile behavior. Failure occurs by tensile splitting with the fractured surface orthogonal to the direction of the maximum tensile stress or strain.

- Prediction of multiaxial behavior
  - In general the material properties are known from simple tests such as uniaxial loadings giving $f_c^{'}$ and f_t . Prediction involves strength calculation in multiaxial situations given the data from the uniaxial tests.
  - In the field of concrete research attempts have been made to apply some of the classical failure theories to concrete. These theories were altered to overcome some disadvantages or otherwise improve their agreement with the phenomenological behavior of concrete. New failure theories were therefore formed.

- Principal stresses:





- Some classical failure theories
  - Maximum principal stress theory
  - Maximum principal strain theory
  - Maximum shear stress theory
  - Internal friction theory
  - Maximum strain energy theory
  - Distortion energy theory
  - Fracture mechanics based theories – stress intensity, toughness – fracture energy release.

  → These introduce either limitations or contradictions when applied to concrete. Modifications to concrete have resulted:

  - Internal friction-maximum stress theory
  - Octahedral shear-normal stress theory
  - Newman's two-part criterion
  - Local deformation theories, etc.

  → Extensive research has been conducted to develop better theories: Elastic-plastic, plastic-fracturing, endochronic, bounding surface etc. approaches.

- Maximum principal stress theory (elastic behavior)
  - $\sigma_1 > \sigma_2 > \sigma_3$
  - Failure occurs when:

    $\sigma_1^{\max} = \sigma_t$ (Tensile strength)

    $\sigma_3^{\max} = \sigma_c = f_c^{'}$ (Compressive strength)

    → It does not reflect splitting nature of failure.

- Maximum principal strain theory (elastic behavior)
  - Failure occurs when:

    $\varepsilon^{\max} = \varepsilon_{\text{limit}} = \varepsilon_t$

- Maximum shear stress theory
  - $\sigma_1 > \sigma_2 > \sigma_3$
  - Failure occurs when:

    $\sigma_1 - \sigma_3 + \lambda(\sigma_1 + \sigma_3) = 2\sigma_s$

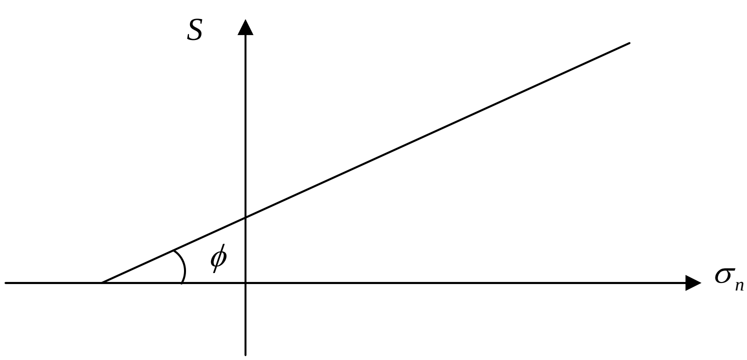
    where $\sigma_1 - \sigma_3$ = shear stress,

    $\lambda(\sigma_1 + \sigma_3)$ = portion of the volumetric stress,

    σ_s = a critical shear stress value (e.g. under pure shear)
  - For metals, $\lambda \cong 0$. For brittle materials, $\lambda \neq 0$.
  - For $\lambda = 0$, the failure criterion becomes

    $\frac{\sigma_1 - \sigma_3}{2} = \sigma_s$
  - The theory gives equal uniaxial tensile and compressive strengths. It is also independent of intermediate stress σ_2. (pressure sensitivity)

- Internal friction theory



  - Consider the effect of normal stress on shear strength:

    $S = K - \tan\phi\sigma_n$

    where S = shear strength,

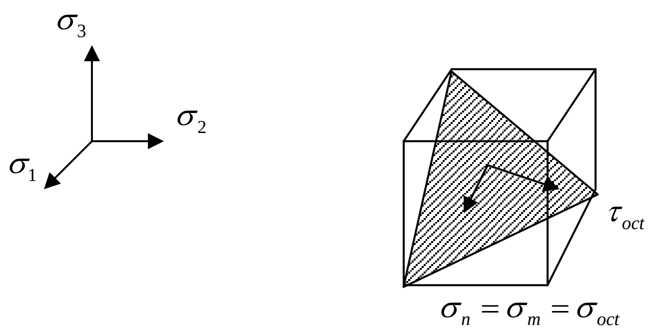
    K = cohesive strength,

    ϕ = angle of internal friction, and

    σ_n = normal stress.
  - Compression increases S and tension decreases S.

- Mohr's theory (generalization of internal friction theory)
  - $S = \tau = f(\sigma)$
  - σ_2 has no effect on strength.
  - $f(\sigma)$ is the envelop of all the circles corresponding to the various states of stress at which failure takes place.

- Octahedral shear and normal stress theory



  - $\sigma_1 > \sigma_2 > \sigma_3$
  - Failure occurs when the octahedral stress exceeds a limiting value.

    $$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

    In uniaxial tension and compression, $\tau_{oct} = \frac{\sqrt{2}}{3}\sigma_i$, i = 1, 2, 3
  - The failure criterion provided by octahedral shear stress theory:

    $$\tau_{oct} = \tau_{\text{limit}} \Rightarrow (\tau_{\text{limit}})_{tension} = (\tau_{\text{limit}})_{compression}$$

    - This gives the same ultimate strength for uniaxial tension and compression. → It is not valid for concrete.
    - Inclusion of $\sigma_{oct} = \sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ improves the prediction.

  - Bresler, Pister tests

$$\frac{\tau_a}{f_c^{'}} = k_1 + k_2\left(\frac{\sigma_a}{f_c^{'}}\right) + k_3\left(\frac{\sigma_a}{f_c^{'}}\right)^2$$

where $\tau_a = \frac{1}{\sqrt{15}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$ and

$$\sigma_a = \sigma_{oct} = \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)$$

- Invariant formulation
  - A failure criterion should be based upon an invariant function of the state of stress, i.e., independent of the choice of the coordinate systems.
  - Stress invariants

    $I_1 = \sigma_1+\sigma_2+\sigma_3$, (more suitable for applying to concrete)

    $I_2 = \sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1$,

    $I_3 = \sigma_1\sigma_2\sigma_3$

    where σ_1, σ_2, σ_3 are principal stresses.

- General stress state representation
  - $\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$ or $\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \sigma_{ij}$, $\sigma_{ij} = \sigma_{ji}$ $\forall\, i, j = 1, 2, 3$

    → $\sigma_{11} = \sigma_1$, $\sigma_{22} = \sigma_2$, $\sigma_{33} = \sigma_3$
  - $\sigma_m = \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}) = \frac{1}{3}\sigma_{ii} = \frac{1}{3}I_1$
  - Average normal stress: $\sigma_a = \frac{1}{3}I_1$
  - Average shear stress: $\tau_a = \sqrt{\frac{2}{15}}\left[I_1^2 - 3I_2\right]^{1/2}$

- Deviatoric stress

  - $S_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$, δ_{ij} = Kronecker's delta $\left(\delta_{ij} = \begin{cases} 1, & \forall i = j \\ 0, & \forall i \neq j \end{cases} \right)$

    $S_{11} = \sigma_{11} - \sigma_m$ $\quad S_{22} = \sigma_{22} - \sigma_m$ $\quad S_{33} = \sigma_{33} - \sigma_m$

    $S_{12} = \sigma_{12}$, $\quad S_{21} = \sigma_{21}$, $\quad S_{31} = \sigma_{31}$

    $S_{13} = \sigma_{13}$ $\quad S_{23} = \sigma_{23}$ $\quad S_{32} = \sigma_{32}$

    where S_{11}, S_{22}, S_{33} are principal stresses.

  - Discussion of physical meaning of deviatoric and hydrostatic stresses.

- Deviatoric stress invariants

  - $J_1 = S_{ij} = S_{11} + S_{22} + S_{33}$

    $$J_2 = \frac{1}{2} S_{ij} S_{ji} = \frac{1}{2}\left(S_{11}{}^2 + S_{22}{}^2 + S_{33}{}^2 \right)$$

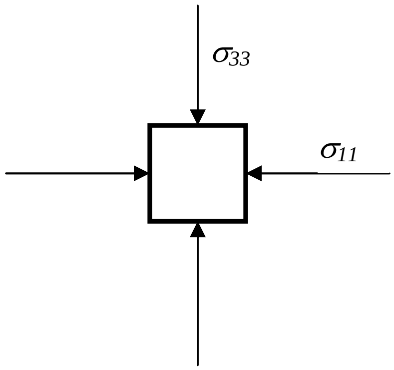
    $$J_3 = \frac{1}{3} S_{ij} S_{jk} S_{ki} = \frac{1}{3}\left(S_{11}{}^3 + S_{22}{}^3 + S_{33}{}^3 \right)$$

- Biaxial loading $(\sigma_{22} = 0)$

  - $S_{11} = \frac{1}{3}(2\sigma_{11} - \sigma_{33})$

    $S_{22} = -\frac{1}{3}(2\sigma_{11} + \sigma_{33})$

    $S_{33} = \frac{1}{3}(2\sigma_{33} - \sigma_{11})$



  - $I_1 = \sigma_{11} + \sigma_{33}$

    $$J_2 = \frac{1}{2} \cdot \frac{1}{9}\left[(2\sigma_{11} - \sigma_{33})^2 + (\sigma_{11} + \sigma_{33})^2 + (2\sigma_{33} - \sigma_{11})^2 \right]$$

    → In general, stress invariants I_1, J_2 are used to characterize the behavior of concrete structures.

- Invariant formulation of the concrete failure

- $F\left(I_1, J_2\right)=0$
- Failure criteria –

  $3J_2+\sigma_3 I_1+\frac{I_1^{\ 2}}{5}=\frac{\sigma_c^{\ 2}}{9}$ and

  $\frac{K^2}{3}J_2-\frac{K^2}{36}I_1 \pm \frac{1}{2}I_1+\frac{1}{3}A_u I_1=\tau_u^{\ 2}$, where A_u, τ_u are material constants.

Multiaxial failure criterion

- Principal stresses based

  $F\left(\sigma_1, \sigma_2, \sigma_3\right)=0$
- Stress invariants based

  $F\left(I_1, J_2, J_3\right)=0$
- One model considering the effect of all the three stress invariants and possessing the observed features of the failure surface such as smoothness, symmetry, convexity, and curved meridians is provided.

$$f\left(\sigma_m, \tau_m, \theta\right)=\frac{1}{\bar{r}\left(\sigma_m, \theta\right)}\frac{\tau_m}{f_c^{'}}-1=0$$

where $\tau_m^2=\frac{3}{5}\tau_{oct}^2=\frac{2}{5}J_2$,

$$\sigma_m=\frac{1}{3}I_1,$$

$$\bar{r}\left(\sigma_m, \theta\right)=\frac{1}{\sqrt{5}f_c^{'}}r\left(\sigma_m, \theta\right)$$

$$r\left(\sigma_m, \theta\right)=\frac{2r_c\left(r_c^2-r_t^2\right)\cos\theta+r_c\left(2r_t-r_c\right)\sqrt{4\left(r_c^2-r_t^2\right)\cos^2\theta+5r_t^2-4r_t r_c}}{4\left(r_c^2-r_t^2\right)\cos^2\theta+\left(r_c-2r_t\right)^2}$$

$$\cos\theta=\frac{2\sigma_1-\sigma_2-\sigma_3}{\sqrt{2\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right]}}$$

$$\frac{r_t}{\sqrt{5}f_c^{'}}=a_0+a_1\frac{\sigma_m}{f_c^{'}}+a_2\left(\frac{\sigma_m}{f_c^{'}}\right)^2 \text{ at } \theta=0^{\circ}$$

$$\frac{r_c}{\sqrt{5}f_c^{'}} = b_0 + b_1\frac{\sigma_m}{f_c^{'}} + b_2\left(\frac{\sigma_m}{f_c^{'}}\right)^2 \text{ at } \theta = 60^\circ$$

- In the tension-biaxial compression zone, the tensile strength is

$$\sigma_{3c} = f_t^{'}\left[1 - \frac{2}{3}\frac{\sigma_1}{1.5f_c^{'}}\right]\left[1 - \frac{2}{3}\frac{\sigma_2}{f_c^{'}}\right]$$

- In the triaxial tension zone, the failure is defined as

$$\sigma_{ic} = f_t^{'} \quad \forall i = 1,2,3$$

- In the compression-biaxial tension zone, the failure is defined as

$$\sigma_{1c} = f_c^{'}$$

$$\sigma_{2c} = f_t^{'}\left[1 - \frac{2}{3}\frac{\sigma_1}{f_c^{'}}\right] = \sigma_{3c}$$

Damage model

- Incremental damage

$$dK = \frac{d\gamma_0^p}{F(I_1,\theta)}$$

where γ_0^p = plastic component of octahedral shear strain,

I_1 = volumetric stress invariant, and

$F(\sigma_{ij}, K_{max}) = 0$ → bounding surface.

- $D = \frac{r}{R}$

where r = current stress vector (distance), and R = distance to bounding surface.

When D = 1, the material is assumed to have failed.

- Constitutive modeling of concrete
  - Approaches for defining stress-strain behavior of concrete:
    - Linear and nonlinear elasticity theories
    - Elastic perfectly plastic models
    - Elastic strain hardening plasticity models
    - Plastic damage (fracturing)-type models
    - Endochronic theory of inelasticity

  - Isotropic stress model
    - The stress-strain law

      $\sigma_{oct} = 3K_S(\varepsilon_{oct})\varepsilon_{oct}$

      $\tau_{oct} = G_S(\gamma_{oct})\gamma_{oct}$

      where σ_{oct} = octahedral normal stress,

      ε_{oct} = octahedral normal strain,

      K_S = secant bulk modulus,

      τ_{oct} = octahedral shear stress,

      γ_{oct} = octahedral shear strain, and

      G_S = secant shear modulus.
    - The nondimensional secant bulk and shear moduli are approximated by

      $$\frac{K_S}{K_0} = ab^{-\varepsilon_{oct}/c} + d$$

      $$\frac{G_S}{G_0} = pq^{-\gamma_{oct}/r} - s\gamma_{oct} + t$$
    - The tangent bulk and shear moduli are

      $$\frac{K_T}{K_0} = a\left[1 - \frac{(\ln b)\varepsilon_{oct}}{c}\right]b^{-\varepsilon_{oct}/c} + d$$

$$\frac{G_T}{G_0} = p\left[1 - \frac{(\ln q)\gamma_{oct}}{c}\right] q^{-\gamma_{oct}/r} + t$$

- The elastic material stiffness matrix:

$$D = \begin{bmatrix} K+4/3G & K-2/3G & K-2/3G & 0 & 0 & 0 \\ & K+4/3G & K-2/3G & 0 & 0 & 0 \\ & & K+4/3G & 0 & 0 & 0 \\ & & & G & 0 & 0 \\ & & & & G & 0 \\ sym. & & & & & G \end{bmatrix}$$

(Young's modulus: $E = \frac{9KG}{3K+G}$, Poisson's ratio: $\nu = \frac{3K-2G}{2(3K+G)}$)

- Isotropic strain model
  - Nonlinear isotropic elastic model
  - The nondimensional secant bulk and shear moduli:

$$\frac{K_S}{K_0} = \frac{1}{1+0.52\left(\sigma_{oct}/f_c^{'}\right)^{1.09}}$$

$$\frac{G_S}{G_0} = \frac{2}{1+3.57\left(\tau_{oct}/f_c^{'}\right)^{1.7}}$$

  - The tangent bulk and shear moduli:

$$\frac{K_T}{K_0} = \frac{1}{1+1.08\left(\sigma_{oct}/f_c^{'}\right)^{1.09}}$$

$$\frac{G_T}{G_0} = \frac{2}{1+9.63\left(\tau_{oct}/f_c^{'}\right)^{1.7}}$$

- Orthotropic model
  - The concept of equivalent uniaxial strains
  - The constitutive law in terms of the material stiffness tensor D_{ijkl}

$$d\sigma_{ij} = D_{ijkl} d\varepsilon_{kl}$$

where $d\sigma_{ij}$ = the tensor of incremental stresses, and

$d\varepsilon_{ij}$ = the tensor of incremental strains.

- The matrix of the tangent stiffness tensor:

$$D = \begin{bmatrix} (1-\nu^2)E_1 & \nu(1+\nu)\sqrt{E_1E_2} & \nu(1+\nu)\sqrt{E_1E_3} & 0 & 0 & 0 \\ & (1-\nu^2)E_2 & \nu(1+\nu)\sqrt{E_2E_3} & 0 & 0 & 0 \\ & & (1-\nu^2)E_3 & 0 & 0 & 0 \\ & & & \phi G_{12} & 0 & 0 \\ & & & & \phi G_{23} & 0 \\ sym. & & & & & \phi G_{31} \end{bmatrix}$$

where E_1, E_2, E_3 = tangent Young's moduli in directions 1, 2, and 3,

$\phi = 1 - 3\nu^2 - 2\nu^3$

G_{12}, G_{23}, G_{31} = incremental shear moduli for planes parallel to coordinates 1-2, 2-3, and 3-1. $G_{12} = \alpha\sqrt{E_1E_2}$, $G_{23} = \alpha\sqrt{E_2E_3}$, $G_{31} = \alpha\sqrt{E_3E_1}$. For uncracked concrete, $\alpha = \frac{1}{2}(1+\nu)$.

- The equivalent uniaxial strain ε_{iu}

$$\varepsilon_{iu} = \frac{\varepsilon_i}{1 - \nu \dfrac{\sigma_j + \sigma_k}{\sigma_i}}$$

where ε_i = principal strain in direction *i*.

- Elastic-hardening plasticity model
  - It is developed for short-term monotonic compressive loading of concrete.
  - The constitutive relationships feature such characteristics of concrete deformational behavior as inelastic dilatancy and frictional effect and inelastic shear caused by hydrostatic pressure (hydrostatic pressure sensitivity).

- Following the incremental theory of plasticity, the total strain increments are

  $\{d\varepsilon\} = \{d\varepsilon^e\} + \{d\varepsilon^p\}$ or $d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p$

  where $d\varepsilon_{ij} = C_{ijkl}^e d\sigma_{kl}$

  $C_{ijkl}^e = \frac{1}{2G}\delta_{ik}\delta_{jl} + \left(\frac{1}{9K} - \frac{1}{6G}\right)\delta_{ij}\delta_{kl}$ = the elastic compliance tensor

- The equation of plastic flow provides

  $d\varepsilon_{ij}^e = \frac{1}{R}\left[\frac{\partial F}{\partial \sigma_{kl}} d\sigma_{kl}\right]\left(\xi_{ij} - \frac{1}{3}\xi_{ii}\right)$ (as the incremental form of elastic strain)

  $d\varepsilon_{ij}^p = \frac{1}{R}\left[\frac{\partial F}{\partial \sigma_{kl}} d\sigma_{kl}\right]\xi_{ij}$ (as the incremental form of plastic strain)

  where R and ξ_{ij} depend on the loading history, and

  F = yield function.

- Plasticity based model

  - $f = 3\sqrt{3J_2 + \bar{\sigma} I_1 + \frac{I_1^2}{5}} = \bar{\sigma}$, where $\bar{\sigma}$ = equivalent stress.

    $d\bar{\sigma} = H' \cdot d\bar{\varepsilon}^P$

    $\frac{\partial f}{\partial \sigma}\{d\sigma\} + \frac{\partial f}{\partial \bar{\bar{\sigma}}} d\bar{\sigma} = d\bar{\sigma}$

    $\Rightarrow \frac{\partial f}{\partial \sigma}\{d\sigma\} + \frac{3I_1}{2\bar{\bar{\sigma}}} d\bar{\sigma} = d\bar{\sigma}$

    $\Rightarrow \frac{\partial f}{\partial \sigma}\{d\sigma\} = \left(1 - \frac{3I_1}{2\bar{\bar{\sigma}}}\right) \cdot H' \cdot d\bar{\varepsilon}^P$

- Hardening law (Flow rule) –

  $\{d\varepsilon^P\} = d\bar{\varepsilon}^P \cdot \left\{\frac{\partial f}{\partial \sigma}\right\}$

→ If the shape of the curve is assumed to expand uniformly in all directions, the flow rule is referred to as the isotropic hardening law.

- Stress increment –

  $\{d\sigma\} = [C]\cdot\{d\varepsilon^e\}$

  $\{d\varepsilon\} = \{d\varepsilon^e\} + \{d\varepsilon^p\}$ = incremental total strain

  $[C]$ = elastic strain to stress transformation matrix

  $\{d\varepsilon^e\} = \{d\varepsilon\} - \{d\varepsilon^p\}$ = elastic strain increment

  $\{d\varepsilon^p\}$ = plastic strain increment

  therefore,

$$\{d\sigma\} = \left\{ [C] - \frac{[C]\cdot\left\{\frac{\partial f}{\partial \sigma}\right\}\cdot\left[\frac{\partial f}{\partial \sigma}\right]\cdot[C]}{\left(1-\frac{3I_1}{2\bar{\sigma}}\right)\cdot H' + \left[\frac{\partial f}{\partial \sigma}\right]\cdot[C]\cdot\left\{\frac{\partial f}{\partial \sigma}\right\}} \right\}\cdot\{d\varepsilon\}$$

  Note that for perfect plasticity, $H' = 0$, this formulation, leading to a non-singular $[C]$, does not cause any numerical difficulty.

Nonlinear analysis of reinforced concrete

- Early studies, by necessity, concentrated on the behavior of isolated elements such as beams, columns, joints, etc. As facilities developed and computing capability expanded the scope broadened to include entire systems such as slabs and beams, coupled shear walls, folded plates, and shells complete with supporting beams for example. This broadening stems both from
  - the desire for better understanding of the behavior of the complete system with the possibility of therefore achieving better structural efficiency, and

  - the need because of increased complexity of the problems requiring solution coupled with more severe demands being placed on the structure.

- Factors which complicate the analysis of reinforcement concrete structures
  - Analytical procedures which may accurately determine stress and deformation states in reinforced concrete members and structures are complicated due to many factors:
    1. Non-linear stress-strain relation
    2. Progressive cracking
    3. Consideration of steel reinforcement
    4. Creep and shrinkage (time-dependent behavior)
    5. Special problems (shear transfer, cyclic loading)

    → The development of finite element method permits realistic evaluation of internal stresses and displacements on which the limit requirements may be based for improved structural efficiency. Furthermore, such refined analytical solutions help in understanding and interpreting the observed behavior of structural elements from experiments.