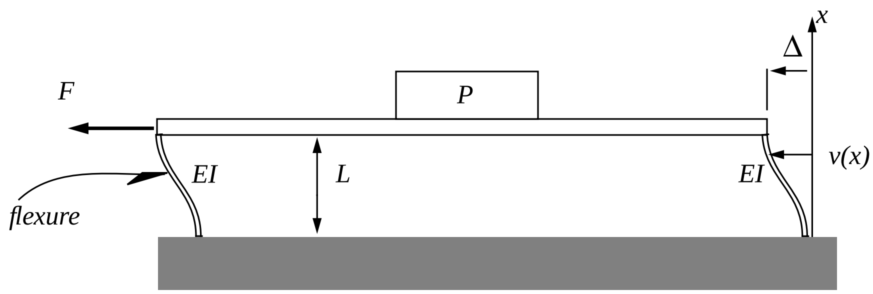
Design Exercise 6
1.050 Solid Mechanics
Fall 2004

You are to design a new experiment for 1.105, one which will demonstrate the effect of an axial load on the bending stiffness of a beam constrained as shown in the appendix. The proposed experimental set-up is shown in the figure.



The beam is actually two (or four) identical “flexures” whose ends, firmly fixed to the rigid plate at some distance *L* above the “ground” may still move freely in the horizontal direction.

Specifications are:

- Maximum values for the weight *P* and the force *F* are limited to the maximum weights we used in our experiments this semester in 1.105.
- The horizontal deflection Δ should be visible, on the order of *0.1 inches*. A dial gage will be used to obtain exact values.
- The apparatus should fit within the area available at a bay in the lab.
- A reduction in stiffness of at least 50% (from the *P=0* value) should be possible.
- The flexures are to be made of a high strength steel and should not yield.

Tensile Strength = 140E03 psi

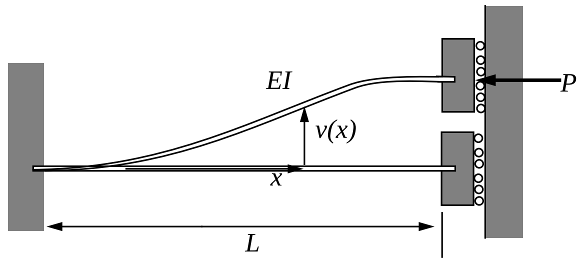
Yield Strength = 80E03 psi

The attached catalogue sheet shows a range of stock sizes for flexure material.[1]

1. http://www1.mscdirect.com/cgi/nnsrhm

Appendix

In class last week, we set-up and solved the buckling problem of a beam fixed at one end - $x = 0$ in the figure - and constrained, in what at first appears to be a very unlikely condition, at the other end $x = L$, namely, the transverse deflection is unrestrained while the slope of the deflected curve is constrained to be zero. The first of these conditions implies that the shear force is zero at $x = L$.



We solved the homogeneous differential equation

$$(EI) \cdot \frac{d^4 v}{dx^4} + P \cdot \frac{d^2 v}{dx^2} = 0$$

with the homogeneous boundary conditions:

$$v(0) = 0 \qquad \left.\frac{dv}{dx}\right|_{x=0} = 0 \qquad at\ x=0$$

$$and$$

$$\left.\frac{dv}{dx}\right|_{x=L} = 0 \qquad (EI) \cdot \frac{d^3 v}{dx^3} + P \cdot \frac{dv}{dx} = -V = 0 \qquad at\ x=L$$

but, since the slope was zero at x = L, this last condition became $\frac{d^3 v}{dx^3} = 0 \quad at\ x=L$.

The general solution to the differential equation is:

$$v(x) = c_1 + c_2 x + c_3 \sin\sqrt{\frac{P}{EI}}x + c_4 \cos\sqrt{\frac{P}{EI}}x$$

and the boundary conditions gave four homogeneous equations for the four constants, the c's, namely:

at $x = 0$.

$$v(0)=0: \quad c_1 \qquad\qquad + c_4 = 0$$
$$dv/dx = 0: \quad c_2 \qquad \lambda c_3 \qquad = 0$$

at $x = L$.

$$dv/dx = 0: \quad c_2 \quad +(\lambda\cos\lambda L)c_3 \quad -(\lambda\sin\lambda L)c_4 = 0$$
$$d^3v/dx^3 = 0: \quad +(\lambda^3\cos\lambda L)c_3 \quad -(\lambda^3\sin\lambda L)c_4 = 0$$

where we set

$$\boxed{\lambda = \sqrt{\frac{P}{EI}}}$$

For a non-trivial solution of this system, the determinant of the coeficients must vanish. This led to the condition

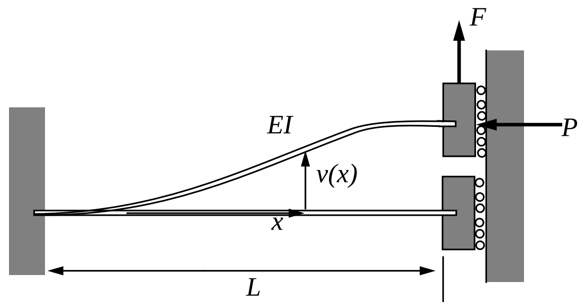
$$\sin\lambda L = 0$$

which gave a lowest buckling load of $\lambda L = \pi$.

The corresponding buckling mode shape was obtained from the boundary conditions, attempting to solve for the c's. From the last condition, with $\sin\lambda L = 0$, we obtained $c_3 = 0$; the third then gave $c_2 = 0$; leaving us with the first which required $c_1 = - c_4$. and hence the mode shape

$$v|_{mode}(x) = c_4 \cdot \left(1 - \cos\frac{\pi x}{L}\right)$$

We now consider the same structural system, but in addition to the axial load P, we apply a transverse force F at the end x = L. The figure below shows our new set-up.



The differential equation remains the same and three of the four boundary conditions remain the same. But the condition at *x=L* on the shear changes for now we have, consistent with our usual sign convention:

$$V = F.$$

So our boundary conditions now are:

$$v(0) = 0 \qquad \left.\frac{dv}{dx}\right|_{x=0} = 0 \qquad at\ x=0$$

and

$$\left.\frac{dv}{dx}\right|_{x=L} = 0 \qquad (EI)\cdot\frac{d^3v}{dx^3} + P\cdot\frac{dv}{dx} = -F \qquad at\ x=L$$

and again, since the slope was zero at x = L, this last condition becomes $\frac{d^3v}{dx^3} = -\frac{F}{EI} \quad at\ x=L$.

This is a significantly different problem now. It is no-longer an eigenvalue problem because we no longer are presented with a homogeneous system of equations. In fact, we can now solve explicitly for the c's from the boundary conditions:

at $x = 0$.

$$v(0)=0: \quad c_1 \qquad\qquad + \ c_4 \quad = 0$$
$$dv/dx = 0: \quad c_2 \qquad \lambda c_3 \qquad = 0$$

at $x = L$.

$$dv/dx = 0: \quad c_2 \quad +(\lambda\cos\lambda L)c_3 \quad -(\lambda\sin\lambda L)c_4 \quad = 0$$
$$d^3v/dx^3 = -F/EI: \quad +(\lambda^3\cos\lambda L)c_3 \quad -(\lambda^3\sin\lambda L)c_4 \quad = F/EI$$

The solution to these, which you are to verify, is:

$$c_1 = \frac{F}{\lambda^3 EI}\cdot\frac{(1-\cos\lambda L)}{\sin\lambda L} \qquad c_2 = -\frac{F}{\lambda^2 EI} \qquad c_3 = \frac{F}{\lambda^3 EI} \qquad and \qquad c_4 = -\frac{F}{\lambda^3 EI}\cdot\frac{(1-\cos\lambda L)}{\sin\lambda L}$$

so the transverse displacement at any x is:

$$v(x) = \frac{F}{\lambda^3 EI}\cdot\left[\frac{(\cos\lambda L - 1)}{\sin\lambda L}\cdot(\cos\lambda x - 1) - \lambda x + \sin\lambda x\right]$$ and

at x = L, the displacement is $v(L) = \Delta = \frac{F}{\lambda^3 EI}\cdot\left[\frac{2(1-\cos\lambda L)}{\sin\lambda L} - \lambda L\right]$ which you should also verify**.

If we let $\boxed{\alpha \equiv \lambda L}$ we can write: $\Delta = \frac{FL^3}{\alpha^3 EI}\cdot\left[\frac{2(1-\cos\alpha)}{\sin\alpha} - \alpha\right]$ **

which, for small alpha - which means for small axial load *P* - I can show that

$$\Delta = \frac{FL^3}{12EI}\cdot[1 + terms\ of\ order\ \alpha^2]$$

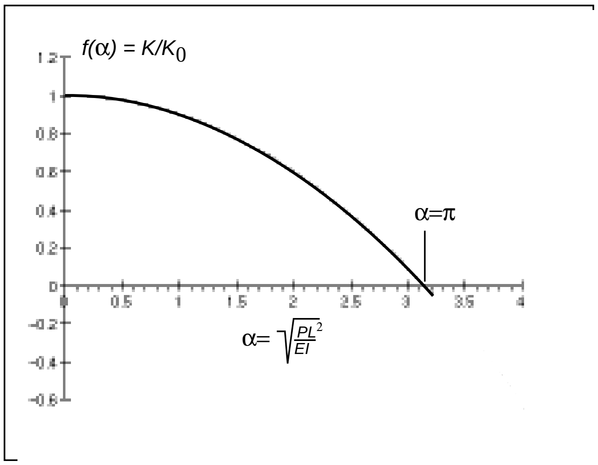
which should look familiar. (Recall Problem 11.1). For alpha *not* small we can write

$$F = K\cdot\Delta \qquad where \qquad K = \frac{12EI}{L^3}\cdot\frac{\alpha^3}{12}\left[\frac{\sin\alpha}{2(1-\cos\alpha) - \alpha\sin\alpha}\right]$$ **

So we see that the stiffness of the system, the *K,* depends upon the axial load *P.* If we let K_0 be the stiffness with no axial load, $P = a = 0$, i.e., $K_0 = 12EI/L^3$ we can write

$$K = K_0 \cdot f(\alpha) \qquad where \qquad f(\alpha) = \frac{\alpha^3}{12}\left[\frac{\sin\alpha}{2(1-\cos\alpha) - \alpha\sin\alpha}\right] \qquad **$$

and plot *f* as a function of alpha.



Note that when $\alpha = \pi$, the stiffness vanishes! This means that it requires no transverse force *F* to produce a transverse displacement! Note too that $\alpha = \pi$ means that the axial load is just equal to the buckling load of the system found previously.

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The bending moment distribution along the flexure is obtained from the moment curvature relationship, knowing *v(x)*.

$$M_b = EI\frac{d^2v}{dx^2}$$

Verify that the bending moment distribution may be written

$$M_b = FL \cdot \left[\frac{\cos\alpha\left(\frac{x}{L}\right) - \cos\alpha\left(1 - \frac{x}{L}\right)}{\alpha\sin\alpha}\right]$$

and that its maximum (in magnitude) is obtained at either end of the beam.