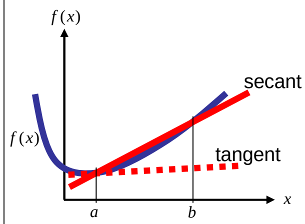
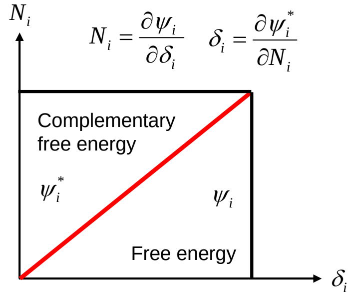
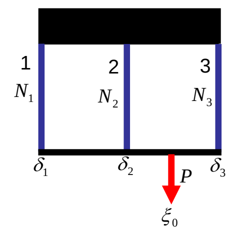
1.050 Engineering Mechanics I

Summary of variables/concepts

Lecture 27 - 37

Variable	Definition	Notes & comments
	$\frac{\partial f}{\partial x}\Big\vert_{x=a}(b-a) \le f(b) - f(a)$	Convexity of a function
W^d	$W^d = \vec{\xi} \cdot \vec{F}^d + \vec{\xi}^d \cdot \vec{R}$	External work
ψ_i^* ψ_i	$N_i = \frac{\partial \psi_i}{\partial \delta_i} \quad \delta_i = \frac{\partial \psi_i^*}{\partial N_i}$  $\sum_i \delta_i N_i = \psi_i^*(N_i) + \psi_i(\delta_i)$	Free energy and complementary free energy 

Lectures 27 and 28: Basic concepts: Convexity, external work, free energy, complementary free energy, introduced initially for truss structures (see schematic show in the lower right part).

Variable	Definition	Notes & comments
Truss problems	$\underbrace{-\left(\psi^{*}-\vec{\xi}^{d}\cdot\vec{R}\right)}_{\text{Complementary energy } =:\, \varepsilon_{\text{com}}} \overset{!}{=} \underbrace{\psi-\vec{\xi}\cdot\vec{F}^{d}}_{\text{Potential energy } =:\, \varepsilon_{\text{pot}}}$ $-\varepsilon_{\text{com}}=\varepsilon_{\text{pot}}$	At elastic solution: Potential energy is equal to negative of complementary energy
	$\underbrace{-\varepsilon_{\text{com}}(N_i^{'},R^{'})}_{\textbf{Lower bound}} \leq \left\{ \begin{array}{c} \max\limits_{N_i^{'}\text{S.A.}}\left(-\varepsilon_{\text{com}}(N_i^{'},R^{'})\right) \\ \text{is equal to} \\ \min\limits_{\delta_i^{'}\text{K.A.}} \varepsilon_{\text{pot}}(\delta_i^{'},\xi_i^{'}) \end{array} \right\} \leq \underbrace{\varepsilon_{\text{pot}}(\delta_i^{'},\xi_i^{'})}_{\textbf{Upper bound}}$	**Upper/lower bound** At the solution to the elasticity problem, the upper and lower bound coincide Consequence of convexity of elastic potentials ψ,ψ^{*}

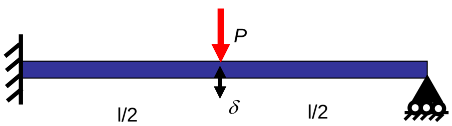
Lectures 27 and 28: Introduction to potential energy and complementary energy, definition **at the** elastic solution, upper/lower bound, example of energy bounds for truss structures. The upper/lower bounds of the expressions are a consequence of the convexity of the elastic potentials (see previous slide).

Variable	Definition	Notes & comments
ψ^*	$\Psi^*(N_i) = \sum_i \frac{1}{2} N_i^2 / K_S$	Complementary free energy (1-D)
ψ	$\Psi(\delta_i) = \sum_i \frac{1}{2} K_S \delta_i^2$	Free energy (1-D)
W, W^*	$W = \sum_{i=1..N} \vec{F}_i^d \cdot \vec{\xi}_i \quad W = \sum_{i=1..N} \vec{R}_i^d \cdot \vec{\xi}_i^d$	Contributions from external work
	$\psi = \frac{1}{2}(W^* + W)$ $\psi^* = \frac{1}{2}(W^* + W)$ $\varepsilon_{\text{pot}} = \frac{1}{2}(W^* - W)$ $\varepsilon_{\text{com}} = \frac{1}{2}(W - W^*)$	Clapeyron's formulas **Significance:** Enables one calculate free energy, complementary free energy, potential energy and complementary energy directly from **the boundary conditions (external work), <u>at the solution</u> ("target")**!

Lectures 27-29: The equations for free energy and complementary free energy for truss structures are summarized. **Lower part: Clapeyron's formulas**, used to calculate the "target" solution, that is, the **results at the solution**. These equations are generally valid, not only for truss structures (but the expressions of how to calculate the individual terms that appear in these equations are different).

Variable	Definition	Notes & comments
$-\varepsilon_{\text{com}}(\underline{\underline{\sigma}}') \le \left\{ \begin{array}{c} \max\limits_{\underline{\underline{\sigma}}'\text{S.A.}} \left(-\varepsilon_{\text{com}}(\underline{\underline{\sigma}}')\right) \\ \text{is equal to} \\ \min\limits_{\vec{\xi}'\text{K.A.}} \varepsilon_{\text{pot}}(\vec{\xi}') \end{array} \right\} \le \varepsilon_{\text{pot}}(\vec{\xi}')$ with $\underline{\underline{\sigma}}'\text{S.A.}$ (lower bound) and $\vec{\xi}'\text{K.A.}$ (upper bound) Lower bound: Complementary energy approach Solution Upper bound: Potential energy approach		Upper/lower bound for 3D elasticity problems
$\varepsilon_{\text{com}}(\underline{\underline{\sigma}}') = \psi^*(\underline{\underline{\sigma}}') - W^*(\vec{T}^d)$ $\varepsilon_{\text{pot}}(\vec{\xi}') = \psi(\underline{\underline{\varepsilon}}') - W(\vec{\xi}^d)$	Displacement contribution $W^*\left(\vec{T}'\right) = \int_{\partial\Omega_{\vec{\xi}^d}} \vec{\xi}^d \cdot \vec{T}'\, da$ $W\left(\vec{\xi}\right) = \int_{\Omega} \vec{\xi} \cdot \rho \vec{g}\, d\Omega + \int_{\partial\Omega_{\vec{T}^d}} \vec{\xi} \cdot \vec{T}^d\, da$ Volume force contribution Stress vector contribution	Complementary energy and potential energy External work contributions
ψ^*	$\psi^* = \int_{\Omega} \frac{1}{2}\left(\frac{\sigma_m^2}{K} + \frac{s^2}{G}\right) d\Omega$ $\sigma_m = \frac{1}{3}\text{trace}(\underline{\underline{\sigma}})$ $\quad s^2 = \frac{1}{2}\left(\underline{\underline{\sigma}} : \underline{\underline{\sigma}} - 3\sigma_m^2\right)$	Complementary free energy (3-D, isotropic material)
ψ	$\psi = \int_{\Omega} \frac{1}{2}\left(K\varepsilon_v^2 + G\varepsilon_d^2\right) d\Omega$ $\varepsilon_v = \text{trace}(\underline{\underline{\varepsilon}})$ $\quad \varepsilon_d^2 = 2\left(\underline{\underline{\varepsilon}} : \underline{\underline{\varepsilon}} - \frac{1}{3}\varepsilon_v^2\right)$	Free energy (3-D, isotropic material)

Lecture 30: Energy bounds for **3D isotropic elasticity**. Note that the external work contribution under force (stress) boundary conditions involves a volume integral due to the volume forces (gravity). The lower part summarizes the equations used to calculate the free energy and complementary free energy, as well as the external work contributions (external work contribution part).

<table>
<tr><th>Variable</th><th>Definition</th><th>Notes & comments</th></tr>
<tr><td>ψ^*</td><td>$\psi^* = \int\limits_{x=0..l} \left[\frac{1}{2}\frac{N^2}{ES} + \frac{1}{2}\frac{M_y^2}{EI} \right] dx$</td><td>Complementary free energy (for beams)</td></tr>
<tr><td>ψ</td><td>$\psi = \int\limits_{x=0..l} \left[\frac{1}{2} ES\left(\varepsilon_{xx}^0\right)^2 + \frac{1}{2} EI\left(\vartheta_y^0\right)^2 \right] dx$</td><td rowspan="2">Free energy (for beams)
Note 1: For 2D, the only contributions are axial forces & moments and axial strains and curvatures
Note 2: Target solution using Clapeyron's formulas</td></tr>
<tr><td colspan="2">

Target solution $\varepsilon_{com} = \frac{1}{2} P\delta$ δ = unknown displacement at point of load application</td></tr>
<tr><td colspan="2">$W^* = \sum_i \left[\vec{\xi}^d(x_i) \cdot \vec{R} + \omega_y(x_i) M_{y,R} \right] = \sum_i \left[\xi_x^d(x_i) R_x + \xi_z^d(x_i) R_z + \omega_y^d(x_i) M_{y,R} \right]$</td><td>External work by prescribed displacements</td></tr>
<tr><td colspan="2">$W = \int\limits_{x=0..l} \vec{\xi}^0 \cdot \vec{f}^d(x) dx + \sum_i \left[\vec{\xi}^0 \cdot \vec{F}^d(x_i) + \omega_y M_y^d(x_i) \right]$
$= \int\limits_{x=0..l} \left[\xi_x^0 f_x^d(x_i) + \xi_z^0 f_z^d(x_i) \right] dx + \sum_i \left[\xi_x^0 F_x^d(x_i) + \xi_z^0 F_z^d(x_i) + \omega_y M_y^d(x_i) \right]$</td><td>External work by prescribed force densities/forces/moments</td></tr>
</table>

Lecture 31: How to calculate free energy, complementary energy and external work for **beam structures**.

<table>
<tr><th>Variable</th><th>Definition</th><th>Notes & comments</th></tr>
<tr><td colspan="2">

$$-\underset{F_x', M_y' \text{ S.A.}}{\varepsilon_{\text{com}}(F_x', M_y')} \le \left\{ \begin{array}{c} \max\limits_{F_x', M_y' \text{ S.A.}} \left(-\varepsilon_{\text{com}}(F_x', M_y')\right) \\ \text{is equal to} \\ \min\limits_{\xi_x', \omega_y' \text{ K.A.}} \varepsilon_{\text{pot}}(\xi_x', \omega_y') \end{array} \right\} \le \underset{\xi_x', \omega_y' \text{ K.A.}}{\varepsilon_{\text{pot}}(\xi_x', \omega_y')}$$

Lower bound
Complementary energy approach
"Stress approach"
Work with unknown but S.A. moments and forces

Solution
F_x', M_y' that provide absolute max of $-\varepsilon_{\text{com}}$
ξ_x', ω_y' that provide absolute min of ε_{pot}

Upper bound
Potential energy approach
"Displacement approach)
Work with unknown but K.A. displacements

</td><td></td></tr>
<tr><td colspan="2">

Step 1: Express target solution (Clapeyron's formulas) – calculate complementary energy AT solution
Step 2: Determine reaction forces and reaction moments
Step 3: Determine force and moment distribution, as a function of reaction forces and reaction moments (need *My* and *N*)
Step 4: Express complementary energy as function of reaction forces and reaction moments (integrate)
Step 5: Minimize complementary energy (take partial derivatives w.r.t. all unknown reaction forces and reaction moments and set to zero); result: set of unknown reaction forces and moments that minimize the complementary energy
Step 6: Calculate complementary energy at the minimum (based on resulting forces and moments obtained in step 5)
Step 7: Make comparison with target solution = find solution displacement

</td><td>Step-by-step procedure – how to solve beam problems with complementary energy approach</td></tr>
</table>

Lectures 31-32: How to solve beam problems using the complementary approach. This slide shows the overview over the upper/lower bounds. The lower part summarizes a step by step procedure of how to solve statically indeterminate beam problems with a complementary energy approach.

<table>
<tr><th>Variable</th><th>Definition</th><th>Notes & comments</th></tr>
<tr><td colspan="2">

- For any homogeneous beam problem, the minimization of the complementary energy with respect to all hyperstatic forces and moments $X_i = \{R_i, M_{y,r;i}\}$ yields the **solution of the linear elastic beam problem**:

$$\frac{\partial}{\partial X_i}\left(\varepsilon_{com}(X_i)\right) = 0$$

$$\frac{1}{2}\left(W - W^*\right) \equiv \min_{X_i} \varepsilon_{com}(X_i)$$

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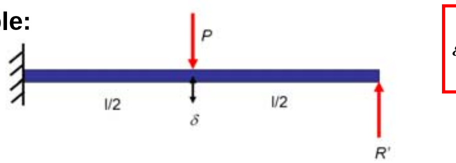
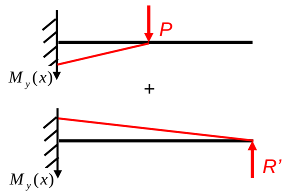
<tr><td colspan="3">

Example:



Hyperstatic force



$$\varepsilon_{com}(R') = \frac{1}{2EI}\left(\frac{l^3}{3}R'^2 - \frac{5}{24}l^3 R' P + \frac{1}{24}l^3 P^2\right)$$

$$\frac{\partial \varepsilon_{com}(R')}{\partial R'} = 0 \qquad R' = \frac{5}{16}P$$

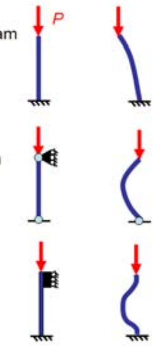
$$\varepsilon_{com}\left(R' = \frac{5}{16}P\right) = \frac{7}{1536\ EI}l^3 P$$

$$\varepsilon_{com} = \frac{1}{2}P\delta \le \varepsilon_{com}\left(R' = \frac{5}{16}P\right) = \frac{7}{1536\ EI}l^3 P$$

$$\delta = \frac{7}{768\ EI}l^3 P$$

</td></tr>
</table>

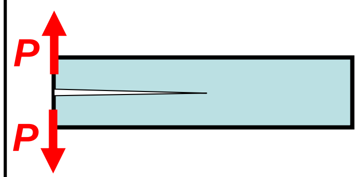
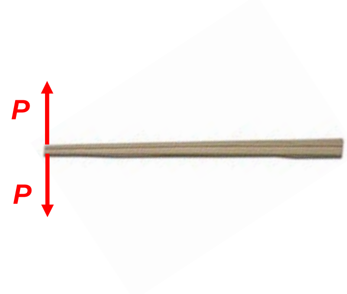
Lectures 31-32: Corollary, how to solve statically indeterminate beam problems using the complementary approach. Summary of the concept that the minimization of the complementary energy with respect to hyperstatic forces and moments provides the exact solution of the linear elastic beam problem.

Variable	Definition	Notes & comments
$P < P_{crit} = \frac{\pi^2 EI}{(el)^2}$ el "effective length"	Clamped cantilever beam $e = 2$ Single supported beam $e = 1$ Double clamped cantilever beam $e = \frac{1}{2}$  	Euler beam buckling Different boundary conditions
 No load applied — $P << P_{crit}$ Small load applied below buckling load Structure stable — $P < P_{crit}$ Large load applied beyonc buckling load Divergence of lateral displacement — $P > P_{crit}$		**Example**: Euler buckling of a frame structure

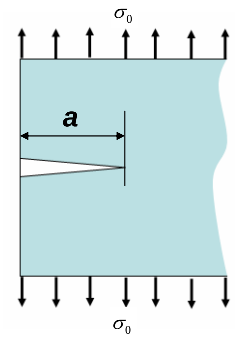
Lectures 33: Buckling of beam structures under compressive load. The lower part summarizes the experiment presented in class.

Variable	Definition	Notes & comments
• (i) iterative solution using conventional small deformation beam theory (**divergence of series**) ↕ • (ii) application of large-deformation beam theory (**nonexistence of solution** since determinant of coefficient matrix is zero - **bifurcation point**) ↕ • (iii) instability is equivalent to **loss of convexity** (energy approach)		Properties and characteristic of instability phenomenon
Images removed due to copyright restrictions: photograph of fault line, World Trade Center towers, shattered wine glass, X-ray of broken bone.		Introduction: Fracture – application and phenomena

Lectures 34: Summary – characteristics of buckling phenomenon (equivalency of divergence of series, nonexistence of solution/bifurcation point/loss of convexity). Introduction to fracture.

Variable	Definition	Notes & comments
$P_{\max} = \sqrt{\dfrac{2\gamma_s bEI}{l^2}}$	 Out-of-plane thickness: *b*	
• Smaller crack length, larger fracture force $P_{\max} \sim \dfrac{1}{l}$ • Larger surface energy, larger fracture force $P_{\max} \sim \sqrt{\gamma_s}$ • Critical load depends on geometry of material (captured in *I*) $P_{\max} \sim \sqrt{bI}$		Useful scaling laws
$G = 2\gamma_s$	$G = -\dfrac{\partial \varepsilon_{pot}}{\partial (lb)}$ $lb = \Gamma$ = unit crack area	**Griffith condition** for crack initiation

Lectures 34 and 35: Fracture mechanics. The most important concept is the Griffith condition. The example on the top summarizes the derivation done in class, representing two beams that are pulled away from each other. This

Variable	Definition	Notes & comments
$G = 1.12^2 \frac{\pi a \sigma_0^2}{E} = 2\gamma$ $\sigma_0 = \sqrt{\frac{2\gamma E}{1.12^2 \pi a}}$		Fracture in a continuum Initial surface crack of length *a*

Lectures 35: Fracture in continuum. The equations summarized in the left side provide the energy release rate *G* for the geometry shown on the right. At the point of fracture, the energy release rate must equal the surface energy. This condition can then be used to determine the critical stress at which the structure begins to fail.