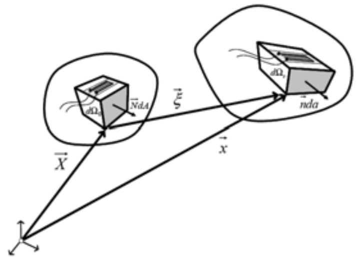
1.050 Engineering Mechanics I

Summary of variables/concepts

Lecture 16-26

Variable	Definition	Notes & comments
		Define undeformed position Deformed position Displacement vector
$\vec{X}$	Position vector, underformed configuration	Note: Distinction between capital “X” and small “x”
$\vec{x}$	Position vector, deformed configuration	
$\vec{\xi}$	$\vec{\xi} = \vec{x} - \vec{X}$	Displacement vector
$\underline{\underline{F}} = F_{ij}\vec{e}_i \otimes \vec{e}_j$	$\underline{\underline{F}} = \text{Grad}(\vec{x}) = \underline{\underline{1}} + \text{Grad}(\vec{\xi})$ $F_{ij} = \frac{\partial x_i}{\partial x_j}$ $d\vec{x} = \underline{\underline{F}} \cdot d\vec{X}$	Deformation gradient tensor Relates position vector of undeformed configuration with deformed configuration

Lectures 16 and 17: Introduction to deformation and strain

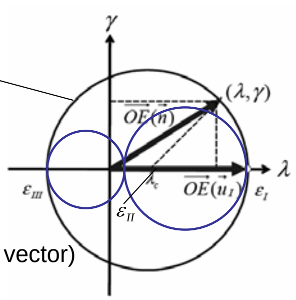
Key concepts: Undeformed and deformed configuration, displacement vector, the transformation between the undeformed and deformed configuration is described by the deformation gradient tensor

Derivation first for general case of large deformation

Variable	Definition	Notes & comments
Large-deformation theory	$J = \frac{d\Omega_d}{d\Omega_0} = \det \underline{\underline{F}}$	J = Jacobian volume change
	$\vec{n}da = J\left(\underline{\underline{F}}^T\right)^{-1} \cdot \vec{N}dA$	Surface change (area & normal)
	$\underline{\underline{E}} = \underline{\underline{F}}^T \underline{\underline{F}} - \underline{\underline{1}} \quad L_d^2 - L_0^2 = d\vec{X} \cdot \left(\underline{\underline{F}}^T \underline{\underline{F}} - \underline{\underline{1}}\right) \cdot d\vec{X} = d\vec{X} \cdot 2\underline{\underline{E}} \cdot d\vec{X}$	Definition of strain tensor
	$\lambda_\alpha = \frac{\Delta L_\alpha}{L_{0,\alpha}} \sqrt{2E_{\alpha\alpha} + 1} - 1$	Relative length variation in the α-direction
	$\sin\theta_{\alpha,\beta} = \frac{2E_{\alpha\beta}}{(1+\lambda_\alpha)(1+\lambda_\beta)}$	Angle change between two vectors
$\underline{\underline{\varepsilon}}$	$\left\|\text{Grad}\,\vec{\xi}\right\| << 1$ $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\text{grad}\,\vec{\xi} + \left(\text{grad}\,\vec{\xi}\right)^T\right)$ $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right)$	Small deformation strain tensor For Cartesian coordinate system

Lecture 18: How to calculate change of geometry (angle, volume, length..)

Small deformation theory: The small deformation theory is valid for small deformations only; for this case the equations simplify. These concepts are most important for the remainder of 1.050.

Variable	Definition	Notes & comments
Small-deformation theory	$\frac{1}{2}\theta(\vec{e}_\alpha,\vec{e}_\beta) = \gamma_{\alpha\beta} = \varepsilon_{\alpha\beta}$ $\frac{1}{2}\theta_{\vec{m},\vec{n}} = \vec{m}\cdot\underline{\underline{\varepsilon}}\cdot\vec{n}$ $\lambda(\vec{e}_\alpha) = \varepsilon_{\alpha\alpha}$ $\lambda_{\vec{n}} = \vec{n}\cdot\underline{\underline{\varepsilon}}\cdot\vec{n}$ $J-1 = \frac{d\Omega_t - d\Omega_0}{d\Omega_0} \simeq \mathrm{tr}\,\underline{\underline{\varepsilon}} = \lambda(\vec{e}_1) + \lambda(\vec{e}_3) + \lambda(\vec{e}_3)$ $\vec{n}\,da \simeq (1+\mathrm{tr}\,\underline{\underline{\varepsilon}})\left(\mathbf{1} - \left(\mathrm{grad}\,\vec{\xi}\right)^T\right)\cdot\vec{N}dA$	Angle change Dilatation Volume change Surface change
	"The" Mohr circle  $\vec{E}(\vec{n}) = \underline{\underline{\varepsilon}}\cdot\vec{n}$ (strain vector) $\vec{E}(\vec{n}) = \lambda\vec{n} + \gamma\vec{t}\begin{cases}\lambda = \vec{n}\cdot\vec{E}(\vec{n}) = \frac{\epsilon_I + \epsilon_{III}}{2} + \frac{\epsilon_I - \epsilon_{III}}{2}\cos 2\vartheta \\ \gamma = \vec{t}\cdot\vec{E}(\vec{n}) = \frac{\epsilon_I - \epsilon_{III}}{2}\sin(-2\vartheta)\end{cases}$	Mohr circle of strain tensor

Lecture 18: Small deformation - Mohr circle for strain tensor. Any strain tensor can be represented in the Mohr plane; this way, one can display a 3D tensor quantity in a 2D projection. All concepts are the same as for the stress tensor Mohr plane. The quantities on the x/y-axes are dilatations and angle change (shear).

Variable	Definition	Notes & comments
δW	Work done by external forces	
$d\psi$	Free energy change	
$d\psi = \delta W$	Non-dissipative deformation= elastic deformation All work done on system stored in free energy	Defines thermodynamics of elastic deformation
$\frac{\partial \psi}{\partial x_i} dx_i = \frac{\partial \psi}{\partial \xi_j} d\xi_j$ $\forall dx_i, \forall d\xi_j$	Solution approach	1D truss systems
$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$ $\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}}$	Link between stress and strain	Also called "generalized Hooke's law"

Lectures 20 and 21: Elasticity, basic definitions. The most important concept of this lecture is that elastic deformation is a thermodynamic process under which no energy dissipation occurs. This concept can be generally applied to characterize any elasticity problem. We derived elasticity for 1D systems (including solution strategy), and then generalized it to 3D. This led to the link between stress and strain.

Variable	Definition	Notes & comments
Isotropic elasticity	Elastic properties of material do NOT depend on direction	Isotropic elasticity described uniquely by 2 parameters, *K* and *G*
$\lvert\underline{\underline{\varepsilon}}\rvert$	$\lvert\underline{\underline{\varepsilon}}\rvert=\sqrt{\frac{1}{2}\left(\underline{\underline{\varepsilon}}:\underline{\underline{\varepsilon}}^T\right)}=\sqrt{\frac{1}{2}\sum_i\sum_j\varepsilon_{ij}^2}$	"Length" of a tensor
$\mathrm{tr}\left(\underline{\underline{\varepsilon}}\right)$	$\mathrm{tr}\left(\underline{\underline{\varepsilon}}\right)=\underline{\underline{\varepsilon}}:\underline{\underline{1}}=\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}=\frac{d\Omega_d-d\Omega_0}{d\Omega_0}$	"Volume change" of a tensor
$\Psi(\varepsilon_v,\varepsilon_d)$	$\Psi=\frac{1}{2}K\varepsilon_v^2+\frac{1}{2}G\varepsilon_d^2$	Free energy due to volume strain and shear strain (assumption, mathematical model to describe elastic behavior of isotropic solids)
$\underline{\underline{\sigma}}=\left(K-\frac{2}{3}G\right)\varepsilon_v\underline{\underline{1}}+2G\underline{\underline{\varepsilon}}=\left(K-\frac{2}{3}G\right)(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})\underline{\underline{1}}+2G\underline{\underline{\varepsilon}}$		Linear isotropic elasticity Tensor notation

Lecture 22: Isotropic elasticity, basic concepts. The most important equation on this slide is the one on the bottom, for linear isotropic elasticity. Note that isotropic elasticity is fully characterized by two constants, *K* and *G*. These two parameters have physical meaning; *K* describes how the free energy changes under volume changes, and *G* describes how the free energy changes under shear (shape) changes.

Variable	Definition	Notes & comments
	$\begin{cases}\sigma_{11}=\left(K-\frac{2}{3}G\right)(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})+2G\varepsilon_{11}\\ \sigma_{22}=\left(K-\frac{2}{3}G\right)(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})+2G\varepsilon_{22}\\ \sigma_{33}=\left(K-\frac{2}{3}G\right)(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})+2G\varepsilon_{33}\\ \sigma_{12}=2G\varepsilon_{12}\\ \sigma_{23}=2G\varepsilon_{23}\\ \sigma_{13}=2G\varepsilon_{13}\end{cases}$	Linear isotropic elasticity Written out for individual stress tensor coefficients
	$\begin{cases}\sigma_{11}=\left(K+\frac{4}{3}G\right)\varepsilon_{11}+\left(K-\frac{2}{3}G\right)\varepsilon_{22}+\left(K-\frac{2}{3}G\right)\varepsilon_{33}\\ \sigma_{22}=\left(K-\frac{2}{3}G\right)\varepsilon_{11}+\left(K+\frac{4}{3}G\right)\varepsilon_{22}+\left(K-\frac{2}{3}G\right)\varepsilon_{33}\\ \sigma_{33}=\left(K-\frac{2}{3}G\right)\varepsilon_{11}+\left(K-\frac{2}{3}G\right)\varepsilon_{22}+\left(K+\frac{4}{3}G\right)\varepsilon_{33}\\ \sigma_{12}=2G\varepsilon_{12}\\ \sigma_{23}=2G\varepsilon_{23}\\ \sigma_{13}=2G\varepsilon_{13}\end{cases}$	Linear isotropic elasticity Written out for individual stress tensor coefficients, collect terms that multiply strain tensor coefficients $c_{1111}=c_{2222}=c_{3333}=K+\frac{4}{3}G$ $c_{1122}=c_{1133}=c_{2233}=K-\frac{2}{3}G$ $c_{1212}=c_{2323}=c_{1313}=2G$

Lecture 22: Isotropic elasticity, equations that relate stress and strain. Here we summarize the equations in different forms. On the bottom, right, you see how to calculate the elasticity tensor coefficients from *K* and *G*.

Overview: 3D linear elasticity

Stress tensor $\underline{\underline{\sigma}}(\vec{x})$

Basis: Physical laws (Newton's laws)

Statically admissible (S.A.)

BCs on boundary of domain Ω

$$\partial\Omega_{\vec{T}^d} : \vec{T}^d(\vec{n}) = \underline{\underline{\sigma}} \cdot \vec{n}$$

$$\Omega : \begin{cases} \vec{T}(\vec{n}) = \underline{\underline{\sigma}} \cdot \vec{n} \\ \operatorname{div} \underline{\underline{\sigma}} + \rho\vec{g} = 0 \\ \sigma_{ij} = \sigma_{ji} \end{cases}$$

Strain tensor $\underline{\underline{\varepsilon}}(\vec{x})$

Basis: Geometry

Kinematically admissible (K.A.)

BCs on boundary of domain Ω

$$\partial\Omega_{\vec{\xi}^d} : \vec{\xi}^d = \vec{\xi}$$

Linear deformation theory

$$\left\| \operatorname{Grad} \vec{\xi} \right\| << 1$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\operatorname{grad} \vec{\xi} + \left(\operatorname{grad} \vec{\xi}\right)^T\right)$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right)$$

Elasticity **Basis**: Thermodynamics

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}} \qquad \sigma_{ij} = c_{ijkl}\varepsilon_{kl} \qquad \underline{\underline{\sigma}} = \left(K - \frac{2}{3}G\right)\varepsilon_v \underline{\underline{1}} + 2G\underline{\underline{\varepsilon}}$$

Isotropic solid

Summary, 3D linear elasticity. This page may be useful to keep an overview over the methods and approaches covered here. This summary is valid for any linear elasticity problem.

Variable	Definition	Notes & comments
Step 1: Write down BCs (stress BCs and displacement BCs), analyze the problem to be solved (read carefully!) **Step 2**: Write governing equations for stress tensor, strain tensor, and constitutive equations that link stress and strain, simplify expressions **Step 3**: Solve governing equations (e.g. by integration), typically results in expression with unknown integration constants **Step 4**: Apply BCs (determine integration constants)		Solution procedure to solve 3D elasticity problems

Lecture 23: Solution approach, 3D isotropic elasticity problems. This is a 4-step solution procedure that guides you through the process.

Variable	Definition	Notes & comments
ε_{xx}	$\varepsilon_{xx} = \varepsilon_{xx}^0 + \vartheta_y^0 z$ $\vartheta_y^0 = -\frac{d^2\xi_z^0}{dx^2}$ Curvature $\varepsilon_{xx}^0 = \frac{d\xi_x^0}{dx}$ Strain $\varepsilon_{xx} = \frac{d\xi_x^0}{dx} - \frac{d^2\xi_z^0}{dx^2} z$	Navier-Bernouilli beam model; strain distribution in beam section
 		Uniaxial beam deformation
ν	$\nu = \frac{1}{2}\frac{3K-2G}{3K+G}$ $\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx}$	Poisson's ratio (lateral contraction under uniaxial tension)
E	$E = \frac{9KG}{3K+G}$ $\sigma_{xx} = E\varepsilon_{xx}$	Young's modulus (relates stresses and strains under uniaxial tension)

Lecture 19 and 24: Beam deformation and beam elasticity. Here we only review the beam bending case for 2D systems. Beam elasticity is a special case of 3D elasticity, adapted for the particular (stretched) geometry of beams. This slide also reviews the introduction of Young's modulus *E* and Poisson's ratio. Both can be calculated from *K* and *G*.

Variable	Definition	Notes & comments
S	$S = \int_S dS$	Cross-sectional area
I	$I = \int_S z^2 dS$	Second order area moment
EI	$M_y = -EI \frac{d^2 \xi_z^0}{dx^2} = EI \vartheta_y$	Beam bending stiffness (relates bending moment and curvature)
$\frac{d^2 \xi_x^0}{dx^2} = -\frac{f_x}{ES}$		Governing differential equation, axial forces
$\frac{d^4 \xi_z^0}{dx^4} = \frac{f_z}{EI}$		Governing differential equation, shear forces
• **Step 1**: Write down BCs (stress BCs and displacement BCs), analyze the problem to be solved (read carefully!) • **Step 2**: Write governing equations for ξ_z, ξ_x ... • **Step 3**: Solve governing equations (e.g. by integration), results in expression with unknown integration constants • **Step 4**: Apply BCs (determine integration constants)		Solution procedure to solve beam elasticity problems

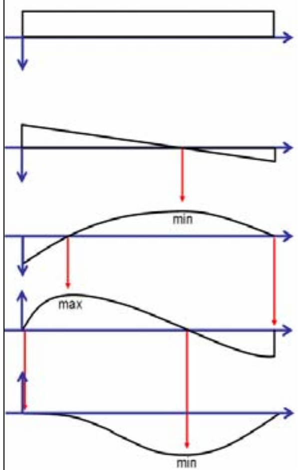
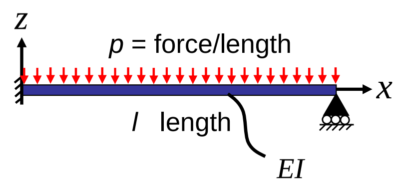
Lecture 25: Beam elasticity, cont'd. Note the two differential equations for axial load/displacement and shear load/displacement in the z-direction. This slide also summarizes the 4-step approach to solve beam problems.

Variable	Definition		Notes & comments
Beam bending (shear forces, moments..)	$\frac{d^4\xi_z}{dx^4} = \frac{f_z}{EI}$	$\frac{d^4\xi_z}{dx^4} EI = f_z$	Shear force density
	$\frac{d^3\xi_z}{dx^3} = -\frac{Q_z}{EI}$	$-\frac{d^3\xi_z}{dx^3} EI = Q_z$	Shear force
	$\frac{d^2\xi_z}{dx^2} = -\frac{M_y}{EI}$	$-\frac{d^2\xi_z}{dx^2} EI = M_y$	Bending moment
	$\frac{d\xi_z}{dx} = -\omega_y$	$-\frac{d\xi_z}{dx} = \omega_y$	Rotation (angle)
	ξ_z	ξ_z	Displacement
Beam stretching (axial forces..)	$\frac{d^2\xi_x^0}{dx^2} = -\frac{f_x}{ES}$	$-ES\frac{d^2\xi_x^0}{dx^2} = f_x$	Axial force density (e.g. gravity)
	$\frac{d\xi_x^0}{dx} = \varepsilon_{xx}^0$		Axial strain
	ξ_x^0		Axial displacement

Lecture 25: Beam elasticity, governing equations for both beam bending and beam stretching. This slide reviews the physical meaning of the different derivatives.

Variable	Definition	Notes & comments
$f(x)$	function of *x*	How to find min/max of functions
$f'(x)=0$	necessary condition for min/max	
$f''(x)<0$	local **maximum**	
$f''(x)>0$	local **minimum**	
$f''(x)=0$	**inflection point**	
• Start from $f_z = EI\xi_z''''$, then work your way up… • Note sign changes: $\xi_z'''' \sim f_z$, $\xi_z''' \sim -Q_z$ ($+\rightarrow-$), $\xi_z'' \sim -M_y$, $\xi_z' \sim -\omega_y$, $\xi_z \sim \xi_z$ ($-\rightarrow+$) • At each level of derivative, first plot extreme cases at ends of beam • Then consider zeros of higher derivatives; determine points of local min/max • ξ_z represents physical shape of the beam ("beam line")		Drawing/sketching approach

Lecture 26: Drawing of beam problems. Note the sign changes, as indicated. The approach is based on the concept of considering min/max values of the functions; since all physical quantities are derivates of one another, this approach can be easily applied to plot the solution.

Variable	Definition	Notes & comments
	$f_z(x) = -p \sim \xi_z''''$ $Q_z(x) = p\left(x - \frac{5}{8}l\right) \sim -\xi_z'''$ $M_y(x) = p\left(\frac{1}{8}l^2 + \frac{x^2}{2} - \frac{5}{8}lx\right) \sim -\xi_z''$ $\omega_y(x) = \frac{p}{EI}\left(\frac{1}{8}l^2 x + \frac{x^3}{6} - \frac{5}{16}lx^2\right) \sim -\xi_z'$ $\xi_z(x) = -\frac{p}{EI}\left(\frac{1}{16}l^2 x^2 + \frac{x^4}{24} - \frac{5}{48}lx^3\right)$	Example 

Lecture 26: Example. Remember to clearly indicate the coordinate system when you draw beam elasticity solutions.

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<tr><th>Variable</th><th>Definition</th><th>Notes & comments</th></tr>
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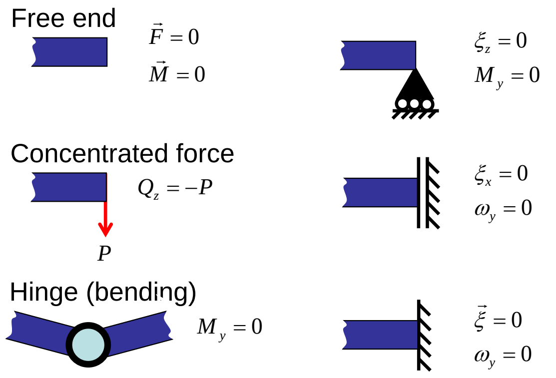

</td><td>Commin beam boundary conditions</td></tr>
<tr><td colspan="2">$\sigma_{xx}(z;x) = E\left(\frac{N(x)}{ES} + \frac{M_y(x)}{EI}z\right) = \frac{N(x)}{S} + \frac{M_y(x)}{I}z$</td><td>Stress distribution within cross-section</td></tr>
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Lecture 26: Common boundary conditions in beam problems, plotting of stress distribution within cross-section.