1.050 Engineering Mechanics I

Review session

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1.050 – Content overview

I. Dimensional analysis

1. On monsters, mice and mushrooms
2. Similarity relations: Important engineering tools

Lectures 1-3
Sept.

II. Stresses and strength

3. Stresses and equilibrium
4. Strength models (how to design structures, foundations.. against mechanical failure)

Lectures 4-15
Sept./Oct.

III. Deformation and strain

5. How strain gages work?
6. How to measure deformation in a 3D structure/material?

Lectures 16-19
Oct.

IV. Elasticity

7. Elasticity model – link stresses and deformation
8. Variational methods in elasticity

Lectures 20-32
Oct./Nov.

V. How things fail – and how to avoid it

9. Elastic instabilities
10. Fracture mechanics
11. Plasticity (permanent deformation)

Lectures 33-37
Dec.

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Notes regarding final exam

- Please contact me or stop by at any time for any questions
- The final will be comprehensive and cover **all material discussed in 1.050**. Note that the last two p-sets will be important for the final.
  - To get an idea about the style of the final, work out old finals and the practice final
  - There will be 2-3 problems with several questions each (e.g. beam problem/ truss problem, continuum problem)
  - We will post old final exams from 2005 and 2006 today
  - We will post an additional, new practice final exam on or around Wednesday next week
  - Another list of variables and concepts will be posted next week
  - **Stay calm, read carefully, and practice time management**

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Stress, strain and elasticity - concepts

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Overview: 3D linear elasticity

Stress tensor $\underline{\underline{\sigma}}(\vec{x})$

Basis: Physical laws (Newton's laws)

Statically admissible (S.A.)

BCs on boundary of domain Ω

$$\partial\Omega_{\vec{T}^d} : \vec{T}^d(\vec{n}) = \underline{\underline{\sigma}} \cdot \vec{n}$$

$$\Omega: \begin{cases} \vec{T}(\vec{n}) = \underline{\underline{\sigma}} \cdot \vec{n} \\ \operatorname{div} \underline{\underline{\sigma}} + \rho\vec{g} = 0 \\ \sigma_{ij} = \sigma_{ji} \end{cases}$$

Strain tensor $\underline{\underline{\varepsilon}}(\vec{x})$

Basis: Geometry

Kinematically admissible (K.A.)

BCs on boundary of domain Ω

$$\partial\Omega_{\vec{\xi}^d} : \vec{\xi}^d = \vec{\xi}$$

Linear deformation theory

$$\left\| \operatorname{Grad} \vec{\xi} \right\| << 1$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\operatorname{grad} \vec{\xi} + \left(\operatorname{grad} \vec{\xi}\right)^T\right)$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right)$$

Elasticity **Basis**: Thermodynamics

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}} \qquad \sigma_{ij} = c_{ijkl}\varepsilon_{kl} \qquad \underline{\underline{\sigma}} = \left(K - \frac{2}{3}G\right)\varepsilon_v \underline{\underline{1}} + 2G\underline{\underline{\varepsilon}}$$

Isotropic solid

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Isotropic elasticity

$$\underline{\underline{\sigma}} = \left(K - \frac{2}{3}G\right)\varepsilon_v \underline{\underline{1}} + 2G\underline{\underline{\varepsilon}} = \left(K - \frac{2}{3}G\right)(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\underline{\underline{1}} + 2G\underline{\underline{\varepsilon}}$$

$$\begin{cases} \sigma_{11} = \left(K - \frac{2}{3}G\right)(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2G\varepsilon_{11} \\ \sigma_{22} = \left(K - \frac{2}{3}G\right)(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2G\varepsilon_{22} \\ \sigma_{33} = \left(K - \frac{2}{3}G\right)(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2G\varepsilon_{33} \\ \sigma_{12} = 2G\varepsilon_{12} \\ \sigma_{23} = 2G\varepsilon_{23} \\ \sigma_{13} = 2G\varepsilon_{13} \end{cases}$$

Linear isotropic elasticity

Written out for individual stress tensor coefficients

$$\begin{cases} \sigma_{11} = \left(K + \frac{4}{3}G\right)\varepsilon_{11} + \left(K - \frac{2}{3}G\right)\varepsilon_{22} + \left(K - \frac{2}{3}G\right)\varepsilon_{33} \\ \sigma_{22} = \left(K - \frac{2}{3}G\right)\varepsilon_{11} + \left(K + \frac{4}{3}G\right)\varepsilon_{22} + \left(K - \frac{2}{3}G\right)\varepsilon_{33} \\ \sigma_{33} = \left(K - \frac{2}{3}G\right)\varepsilon_{11} + \left(K - \frac{2}{3}G\right)\varepsilon_{22} + \left(K + \frac{4}{3}G\right)\varepsilon_{33} \\ \sigma_{12} = 2G\varepsilon_{12} \\ \sigma_{23} = 2G\varepsilon_{23} \\ \sigma_{13} = 2G\varepsilon_{13} \end{cases}$$

Linear isotropic elasticity

Written out for individual stress tensor coefficients, collect terms that multiply strain tensor coefficients

$$c_{1111} = c_{2222} = c_{3333} = K + \frac{4}{3}G$$

$$c_{1122} = c_{1133} = c_{2233} = K - \frac{2}{3}G$$

$$c_{1212} = c_{2323} = c_{1313} = 2G$$

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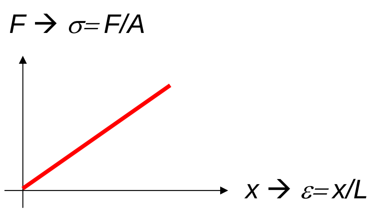
Variable	Definition	Notes & comments
ν	$\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx} \quad \nu = \frac{1}{2}\frac{3K-2G}{3K+G}$	Poisson's ratio (lateral contraction under uniaxial tension)
E	$E = \frac{9KG}{3K+G}$ $\sigma_{xx} = E\varepsilon_{xx}$	Young's modulus (relates stresses and strains under uniaxial tension)
 		Uniaxial beam deformation

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Solving problems with strength approach

Use conditions for S.A. plus strength criterion (S.C.)

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<table>
<tr><th>Variable</th><th>Definition</th><th>Notes & comments</th></tr>
<tr><td>Two pillars of stress-strength approach</td><td>At any point, $\underline{\underline{\sigma}}$ must be:
(1) Statically admissible (S.A.)
and
(2) Strength compatible (S.C.)</td><td></td></tr>
<tr><td></td><td colspan="2">• Equilibrium conditions “only” specify statically admissible stress field, without worrying about if the stresses can actually be sustained by the material – S.A.
From EQ condition for a REV we can integrate up (upscale) to the structural scale
Examples: Many integrations in homework and in class; Hoover dam etc.

• Strength compatibility adds the condition that in addition to S.A., the stress field must be compatible with the strength capacity of the material – S.C.
In other words, at no point in the domain can the stress vector exceed the strength capacity of the material
Examples: Sand pile, foundation etc. – Mohr circle
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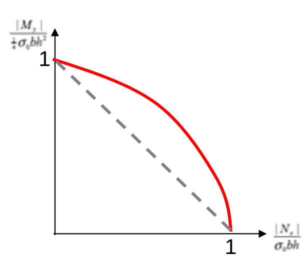
</table>

Variable	Definition	Notes & comments
D_S	$\forall x;\ \left(\overrightarrow{F}_S, \overrightarrow{\mathcal{M}}_S\right) \in D_S(x) \Leftrightarrow f\left(x, \overrightarrow{F}_S(x), \overrightarrow{\mathcal{M}}_S(x)\right) \leq 0$	Strength domain for beams
$\lvert\mathcal{M}_y\rvert_{\lim} = M_0$	$\lvert\mathcal{M}_y\rvert_{\lim} = M_0 = \frac{1}{4}\sigma_0 bh^2$ For rectangular cross-section *b,h*	Moment capacity for beams
$\lvert N_x\rvert_{\lim} = N_0$	$\lvert N_x\rvert_{\lim} = N_0 = bh\sigma_0$	Strength capacity for beams
$f(M_y, N_x) \leq 0$	$f(M_y, N_x) = \frac{\lvert M_y\rvert}{M_0} + \frac{\lvert N_x\rvert}{N_x} - 1 \leq 0$ $f(M_y, N_x) = \frac{\lvert M_y\rvert}{M_0} + \left(\frac{\lvert N_x\rvert}{N_x}\right)^2 - 1 \leq 0$ 	M-N interaction (linear) M-N interaction (actual); convexity 10

Variable	Definition	Notes & comments
Safe strength domain	$\forall i;\ 0 \leq Q^{(i)} \leq \alpha_i Q^{(i)}_{\text{lim}}$ $\sum_{i=1,n} \frac{Q^{(i)}}{Q^{(i)}_{\text{lim}}} \leq \sum_{i=1,n} \alpha_i = 1$ $Q^{(i)}_{\text{lim}}$: load bearing capacity of *i*-th load case	Linear combination is safe (convexity)

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Mohr circle

Display 3D strain tensor in 2D projection – enables us to 'see' largest shear stresses, largest normal stresses…

Thereby facilitates application of strength criterion

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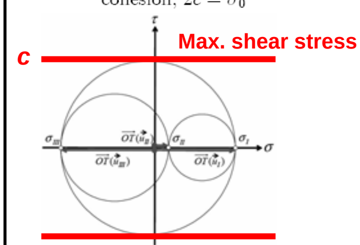
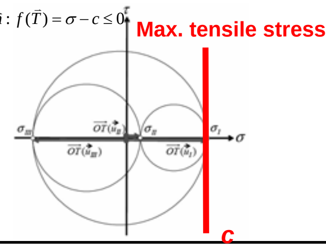
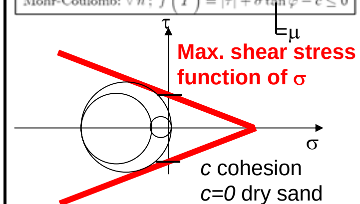
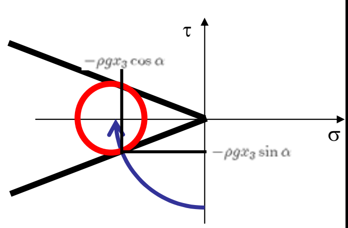
Variable	Definition	Notes & comments
D_k	$\forall \vec{x};\ \boldsymbol{\sigma}(\vec{x}) \in D_k \Leftrightarrow f(\vec{x}, \boldsymbol{\sigma}(\vec{x})) \overset{\text{S.C.}}{\leq} 0$	Strength domain (general definition) Equivalent to condition for S.C.
$D_{k,\text{Tresca}}$	Tresca: $\forall \vec{n};\ f(\vec{T}) = \lvert\tau\rvert - c \leq 0$ cohesion, $2c = \sigma_0$ 	Tresca criterion
$D_{k,\text{Tension-cutoff}}$	$\forall \vec{n}: f(\vec{T}) = \sigma - c \leq 0$ 	Tension cutoff criterion 13

Variable	Definition	Notes & comments
$D_{k,\text{Mohr-Coulomb}}$	Mohr-Coulomb: $\forall \vec{n};\ f(\vec{T}) = \lvert\tau\rvert + \sigma \tan\varphi - c \leq 0$ 	Mohr-Coulomb
$\alpha_{\lim}$	 $\alpha_{\lim} = \varphi$	Angle of repose 14

Variable	Definition	Notes & comments
S	$S = \int_S dS$	Cross-sectional area
I	$I = \int_S z^2 dS$	Second order area moment
EI	$M_y = -EI \frac{d^2\xi_z^0}{dx^2} = EI\vartheta_y$	Beam bending stiffness (relates bending moment and curvature)
$\frac{d^2\xi_x^0}{dx^2} = -\frac{f_x}{ES}$		Governing differential equation, axial forces
$\frac{d^4\xi_z^0}{dx^4} = \frac{f_z}{EI}$		Governing differential equation, shear forces
• **Step 1**: Write down BCs (stress BCs and displacement BCs), analyze the problem to be solved (read carefully!) • **Step 2**: Write governing equations for ξ_z, ξ_x ... • **Step 3**: Solve governing equations (e.g. by integration), results in expression with unknown integration constants • **Step 4**: Apply BCs (determine integration constants)		Solution procedure to solve beam elasticity problems

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Energy approach

Approximate solution or find exact solution

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Upper/lower bounds

$$-\underset{\underline{\underline{\sigma}}'\text{S.A.}}{\varepsilon_{\text{com}}(\underline{\underline{\sigma}}')} \le \left\{ \begin{array}{c} \max\limits_{\underline{\underline{\sigma}}'\text{S.A.}}\left(-\varepsilon_{\text{com}}(\underline{\underline{\sigma}}')\right) \\ \text{is equal to} \\ \min\limits_{\vec{\xi}'\text{K.A.}} \varepsilon_{\text{pot}}(\vec{\xi}') \end{array} \right\} \le \underset{\vec{\xi}'\text{K.A.}}{\varepsilon_{\text{pot}}(\vec{\xi}')}$$

Lower bound

Complementary energy approach

Solution

Upper bound

Potential energy approach

$$\varepsilon_{\text{com}}(\underline{\underline{\sigma}}') = \psi^*(\underline{\underline{\sigma}}') - W^*(\vec{T}')$$

$$\varepsilon_{\text{pot}}(\vec{\xi}') = \psi(\underline{\underline{\varepsilon}}') - W(\vec{\xi}')$$

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Boundary conditions

Important concept in 1.050 and elsewhere

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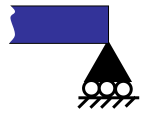
Important BCs in beams/frames

Free end

$\vec{F} = 0$

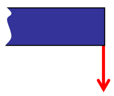
$\vec{M} = 0$



$\xi_z = 0$

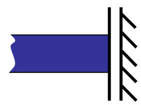
$M_y = 0$

Concentrated force



P

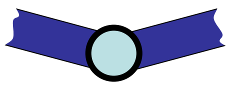
$Q_z = -P$



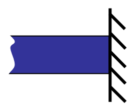
$\xi_x = 0$

$\omega_y = 0$

Hinge (bending)



$M_y = 0$



$\vec{\xi} = 0$

$\omega_y = 0$

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Buckling of beams in compression

Elastic instability

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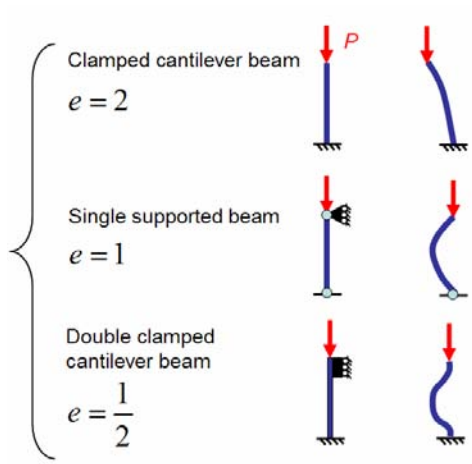
Buckling

$$P < P_{crit} = \frac{\pi^2 EI}{(el)^2}$$

el "effective length"



Euler beam buckling
Different boundary conditions

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Fracture mechanics

How to treat cracks in a continuum

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Example: 3D fracture model

Expressions for G can be found for a variety of geometries and structures:

$\vec{T}^d = \sigma_0 \vec{e}_z$ far away from crack

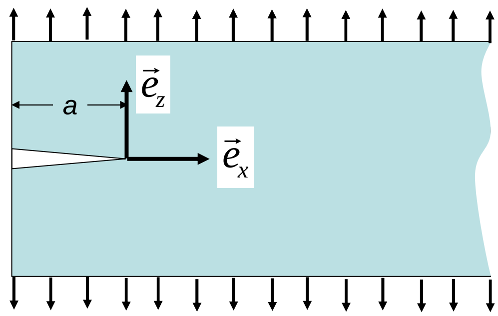
For this geometry:

$$G = 1.12^2 \frac{\pi a \sigma_0^2}{E} \overset{!}{=} 2\gamma_s$$

$$\sigma_0 = \sqrt{\frac{2\gamma_s E}{1.12^2 \pi a}}$$

$$\sigma_0 = \sqrt{\frac{1}{1.12^2 \pi a}} K_{Ic}$$

σ_0



σ_0

"Surface crack"

$\vec{T}^d = -\sigma_0 \vec{e}_z$

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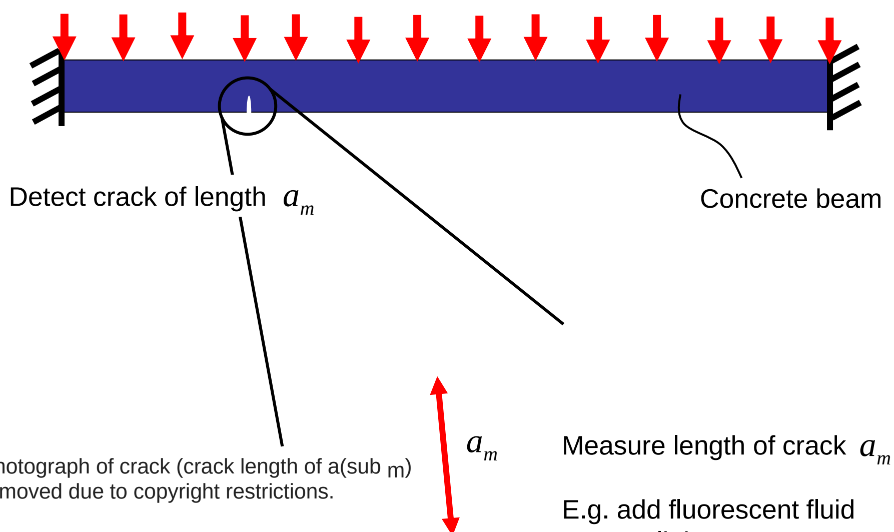
Example application



Photograph of crack (crack length of a(sub m) removed due to copyright restrictions.

Measure length of crack a_m

E.g. add fluorescent fluid use UV light

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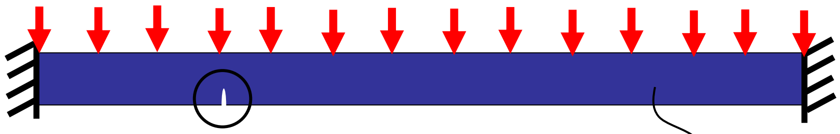
http://www.amesresearch.com/images/cshst/block_before.jpg

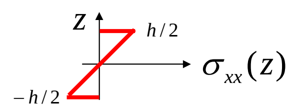
Example application



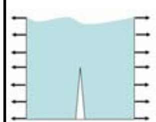
Question: Will structure fail?

Solution: Solving beam problem provides us with stress distribution in section

$$\sigma_{xx}(z;x) = \frac{M_y(x)}{I} z \qquad \sigma_{xx}(z) = \frac{\sigma_0}{h/2} z$$



Calculate critical fracture stress and compare with stress in beam structure



$$\sigma_{0,crit} = \sqrt{\frac{1}{1.12^2 \pi a_m}} K_{Ic} \begin{cases} \sigma_{xx}(z = -\frac{h}{2}) \geq \sigma_{0,crit} & \text{failure} \\ \sigma_{xx}(z = -\frac{h}{2}) < \sigma_{0,crit} & \text{no failure} \end{cases}$$