

# 1.050 Engineering Mechanics I

## Lecture 34

How things fail – and how to avoid it

Additional notes – energy approach

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# 1.050 – Content overview

## I. Dimensional analysis

1. On monsters, mice and mushrooms
2. Similarity relations: Important engineering tools

Lectures 1-3  
Sept.

## II. Stresses and strength

3. Stresses and equilibrium
4. Strength models (how to design structures, foundations.. against mechanical failure)

Lectures 4-15  
Sept./Oct.

## III. Deformation and strain

5. How strain gages work?
6. How to measure deformation in a 3D structure/material?

Lectures 16-19  
Oct.

## IV. Elasticity

7. Elasticity model – link stresses and deformation
8. Variational methods in elasticity

Lectures 20-32  
Oct./Nov.

## V. How things fail – and how to avoid it

9. Elastic instabilities
10. Fracture mechanics
11. Plasticity (permanent deformation)

Lectures 33-37  
Dec.

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# 1.050 – Content overview

## I. Dimensional analysis

## II. Stresses and strength

## III. Deformation and strain

## IV. Elasticity

## V. How things fail – and how to avoid it

Lecture 33 (Mon): Buckling (loss of convexity)

Lecture 34 (Wed): Fracture mechanics I

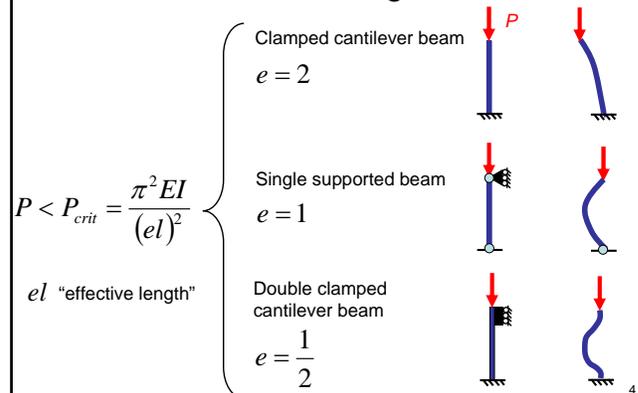
Lecture 35 (Fri – teaching evaluation): Fracture mechanics II

Lecture 36 (Mon): Plastic yield

Lecture 37 (Wed): Wrap-up plastic yield and closure

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## Selection of boundary conditions: Euler buckling



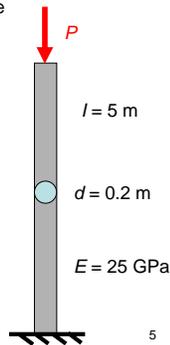
## Example: Concrete column w/ circular cross-section

Equation provides critical buckling load: Force must be below this load to avoid failure

$$P < P_{crit} = \frac{\pi^2 EI}{(2l)^2}$$

$$EI = \frac{E\pi d^4}{64} \quad \text{Circular cross-section}$$

$$P_{crit} = 194 \text{ kN}$$



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## Recall – lecture 33

Analyzed buckling instability (displacement convergence at point where load is applied) using two approaches:

- (i) iterative solution using conventional small deformation beam theory (**divergence of series**)



- (ii) application of large-deformation beam theory (**nonexistence of solution** since determinant of coefficient matrix is zero - **bifurcation point**)



- (iii) instability is equivalent to **loss of convexity** (energy approach)

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## Express potential energy for large deformation beam theory



Governing equations:

$$\frac{dM_y}{dx} = Q_z \quad \text{Conventional, small deformation beam theory}$$

Moment generated by rotation of beam

$$\frac{dM_y}{dX} = Q_z + \omega_y Q_x \quad \text{Large deformation beam theory}$$

Moment generated by rotation of beam

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## Express potential energy for large deformation beam theory

Goal: Show that elastic instability corresponds to a **loss of convexity**

$$\mathcal{E}_{pot} = \mathcal{E}_{pot}(\vartheta_y, \omega_y)$$

New term  
large-deformation beam theory

Note: Potential energy in large-deformation beam theory depends also on rotation (because rotations create moments)

Calculation of potential energy

$$\mathcal{E}_{pot}(\vartheta_y, \omega_y) = \int_l \frac{1}{2} (EI \vartheta_y^2 + F_x \omega_y^2) dX - W(\xi^0, \omega_y) \quad 8$$

## Express potential energy for large deformation beam theory

As studied previously:  $\varepsilon_{pot}(\vartheta_y, \omega_y) \leq \varepsilon_{pot}(\vartheta_y', \omega_y')$

Potential energy at solution      Potential energy at any other K.A. displacement field

### Lower bound approach

Solution corresponds to absolute min of potential energy for K.A. displacement field

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## Express potential energy for large deformation beam theory



Assume K.A. displacement field:  $\xi'_z(X) = \delta f(X/l)$

Parameter

$f(X/l)$  must satisfy BCs...:  $X=0: \begin{cases} \xi'_z = 0 \Rightarrow f(0) = 0 \\ \frac{d\xi'_z}{dX} = 0 \Rightarrow \frac{df}{dX}(0) = 0 \end{cases}$

$f\left(\frac{X}{l}\right) = 3\left(\frac{1}{2}\left(\frac{X}{l}\right)^2 - \frac{1}{6}\left(\frac{X}{l}\right)^3\right)$

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## Express potential energy for large deformation beam theory

$$\varepsilon_{pot}(\vartheta_y, \omega_y) = \int_l \frac{1}{2} (EI \vartheta_y^2 + F_x \omega_y^2) dX - W(\bar{\xi}^0, \omega_y)$$

Now express potential energy as function of parameter  $\delta$ :

$$\varepsilon_{pot} = \frac{1}{2} \delta^2 \int_l \left( EI \left( \frac{d^2 f}{dx^2} \right)^2 - P \left( \frac{df}{dx} \right)^2 \right) dX - P \frac{\Delta}{l} \delta$$

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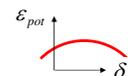
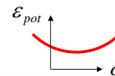
## Express potential energy for large deformation beam theory

Necessary condition for minimum of potential energy:

$$\frac{\partial \varepsilon_{pot}}{\partial \delta} = \delta \int_l \left( EI \left( \frac{d^2 f}{dx^2} \right)^2 - P \left( \frac{df}{dx} \right)^2 \right) dX - P \frac{\Delta}{l} \stackrel{!}{=} 0$$

Also, the expression of  $\varepsilon_{pot}$  must be convex!

Convexity:  $\frac{\partial^2 \varepsilon_{pot}}{\partial \delta^2} > 0$       Loss of convexity:  $\frac{\partial^2 \varepsilon_{pot}}{\partial \delta^2} \leq 0$   
 Elastic instability!



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## Express potential energy for large deformation beam theory

Loss of convexity:  $\frac{\partial^2 \mathcal{E}_{pot}}{\partial \delta^2} \leq 0$

$$\frac{\partial^2 \mathcal{E}_{pot}}{\partial \delta^2} = \int_l \left( EI \left( \frac{d^2 f}{dx^2} \right)^2 - P \left( \frac{df}{dx} \right)^2 \right) dX$$

Instability occurs if...

$$P \geq \frac{\int_l \left( EI \left( \frac{d^2 f}{dx^2} \right)^2 \right) dX}{\int_l \left( \frac{df}{dx} \right)^2 dX} = P_{crit} = EI_{zz} \frac{\int_0^l \left( 3 \frac{l-X}{l^3} \right)^2 dX}{\int_0^l \left( \frac{3}{2} X \frac{2l-X}{l^3} \right)^2 dX} = 2.5 \frac{EI_{zz}}{l^2}$$

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## Express potential energy for large deformation beam theory

Notes:

- Critical load obtained from potential energy approach yields **upper bound** of actual critical buckling load
- Iterations with **first order displacement field** leads to identical series expansion as in the iterative approach with small deformation beam theory

Approximate solution:

$$P_{crit} = 2.5 \frac{EI}{l^2}$$

Actual solution:

$$P_{crit} = 2.5 \frac{EI}{l^2} > P_{crit} = \frac{\pi^2}{4} \frac{EI}{l^2} = 2.4674 \frac{EI}{l^2}$$

## Summary

Analyzed buckling instability (displacement convergence at point where load is applied) using two approaches:

- (i) iterative solution using conventional small deformation beam theory (**divergence of series**)
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## Fracture mechanics

How brittle materials fail  
"crack extension"

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## Brittle fracture

**Brittle fracture** = typically uncontrolled response of a structure, often leading to sudden malfunction of entire system

Images removed due to copyright restrictions: Photograph of fault line, World Trade Center towers, shattered wine glass, X-ray of broken bone.

## Brittle failure – crack extension

Images removed due to copyright restrictions.

Snapshots show microscopic processes as a crack extends.

During crack propagation, elastic energy stored in the material is dissipated<sup>18</sup> by breaking atomic bonds

M.J. Buehler, H. Tang, et al., *Phys. Rev. Letters*, 2007