1.050 Engineering Mechanics I

Lecture 34

How things fail – and how to avoid it

Additional notes – energy approach

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1.050 – Content overview

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1.050 – Content overview

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Selection of boundary conditions: Euler buckling

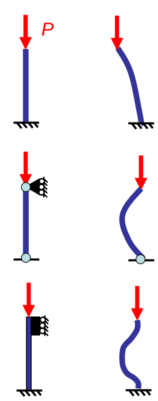
$$P < P_{crit} = \frac{\pi^2 EI}{(el)^2}$$

el "effective length"

Clamped cantilever beam
$e = 2$

Single supported beam
$e = 1$

Double clamped cantilever beam
$e = \frac{1}{2}$



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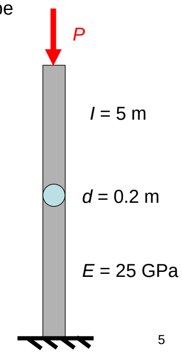
Example: Concrete column w/ circular cross-section

Equation provides critical buckling load: Force must be below this load to avoid failure

$$P < P_{crit} = \frac{\pi^2 EI}{(2l)^2}$$

$$EI = \frac{E\pi d^4}{64}$$ Circular cross-section

$$\boxed{P_{crit} = 194 \text{ kN}}$$



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Recall – lecture 33

Analyzed buckling instability (displacement convergence at point where load is applied) using two approaches:

- (i) iterative solution using conventional small deformation beam theory (**divergence of series**)
- (ii) application of large-deformation beam theory (**nonexistence of solution** since determinant of coefficient matrix is zero - **bifurcation point**)
- (iii) instability is equivalent to **loss of convexity** (energy approach)

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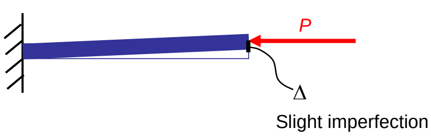
Express potential energy for large deformation beam theory

Reminder:



Governing equations:

$$\frac{dM_y}{dx} = Q_z$$

Conventional, small deformation beam theory
Moment generated by rotation of beam

$$\frac{dM_y}{dX} = Q_z + \omega_y Q_x$$

Large deformation beam theory
Moment generated by rotation of beam

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Express potential energy for large deformation beam theory

Goal: Show that elastic instability corresponds to a **loss of convexity**

$$\varepsilon_{pot} = \varepsilon_{pot}(\vartheta_y, \omega_y)$$

New term
large-deformation beam theory

Note: Potential energy in large-deformation beam theory depends also on rotation (because rotations create moments)

Calculation of potential energy

$$\varepsilon_{pot}(\vartheta_y, \omega_y) = \int_l \frac{1}{2}\left(EI\vartheta_y^2 + F_x\omega_y^2\right)dX - W(\vec{\xi}^0, \omega_y)$$

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Express potential energy for large deformation beam theory

As studied previously: $\varepsilon_{pot}(\vartheta_y, \omega_y) \le \varepsilon_{pot}(\vartheta_y', \omega_y')$

Potential energy at solution

Potential energy at any other K.A. displacement field

Lower bound approach

Solution corresponds to absolute min of potential energy for K.A. displacement field

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Express potential energy for large deformation beam theory

Assume K.A. displacement field: $\xi'_Z(X) = \delta\, f(X/\ell)$

Parameter

$f(X/l)$ must satisfy BCs…: $X = 0: \begin{cases} \xi'_Z = 0 & \Rightarrow\ f(0) = 0 \\ \dfrac{d\xi'_Z}{dX} = 0 & \Rightarrow\ \frac{df}{dX}(0) = 0 \end{cases}$

$$f\left(\frac{X}{\ell}\right) = 3\left(\frac{1}{2}\left(\frac{X}{\ell}\right)^2 - \frac{1}{6}\left(\frac{X}{\ell}\right)^3\right)$$

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Express potential energy for large deformation beam theory

$$\varepsilon_{pot}(\vartheta_y, \omega_y) = \int_l \frac{1}{2}\left(EI\vartheta_y^2 + F_x\omega_y^2\right)dX - W(\vec{\xi}^{\,0}, \omega_y)$$

Now express potential energy as function of parameter δ:

$$\boxed{\varepsilon_{pot} = \frac{1}{2}\delta^2 \int_l \left(EI\left(\frac{d^2 f}{dx^2}\right)^2 - P\left(\frac{df}{dx}\right)^2\right)dX - P\frac{\Delta}{l}\delta}$$

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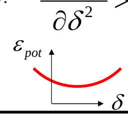
Express potential energy for large deformation beam theory

Necessary condition for minimum of potential energy:

$$\frac{\partial \varepsilon_{pot}}{\partial \delta} = \delta \int_l \left(EI\left(\frac{d^2 f}{dx^2}\right)^2 - P\left(\frac{df}{dx}\right)^2\right)dX - P\frac{\Delta}{l} \overset{!}{=} 0$$

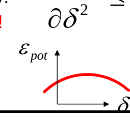
Also, the expression of ε_{pot} must be convex!

Convexity: $\dfrac{\partial^2 \varepsilon_{pot}}{\partial \delta^2} > 0$



Loss of convexity: $\dfrac{\partial^2 \varepsilon_{pot}}{\partial \delta^2} \le 0$

Elastic instability!



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Express potential energy for large deformation beam theory

Loss of convexity: $\frac{\partial^2 \varepsilon_{pot}}{\partial \delta^2} \le 0$

$$\frac{\partial^2 \varepsilon_{pot}}{\partial \delta^2} = \int_l \left(EI \left(\frac{d^2 f}{dx^2} \right)^2 - P \left(\frac{df}{dx} \right)^2 \right) dX$$

Instability occurs if…

$$P \ge \frac{\int_l \left(EI \left(\frac{d^2 f}{dx^2} \right)^2 \right) dX}{\int_l \left(\frac{df}{dx} \right)^2 dX} = P_{crit} = EI_{zz} \frac{\int_0^\ell \left(3\frac{\ell - X}{\ell^3}\right)^2 dX}{\int_0^\ell \left(\frac{3}{2} X \frac{2\ell - X}{\ell^3}\right)^2 dX} = 2.5 \frac{EI_{zz}}{\ell^2}$$

Express potential energy for large deformation beam theory

Notes:

- Critical load obtained from potential energy approach yields **upper bound** of actual critical buckling load
- Iterations with **first order displacement field** leads to identical series expansion as in the iterative approach with small deformation beam theory

Approximate solution:

$$P_{crit} = 2.5 \frac{EI}{l^2}$$

$>$

Actual solution:

$$P_{crit} = \frac{\pi^2}{4} \frac{EI}{l^2} = 2.4674 \frac{EI}{l^2}$$

Summary

Analyzed buckling instability (displacement convergence at point where load is applied) using two approaches:

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Fracture mechanics

How brittle materials fail

"crack extension"

Brittle fracture

Brittle fracture = typically uncontrolled response of a structure, often leading to sudden malfunction of entire system

Images removed due to copyright restrictions: Photograph of fault line,
World Trade Center towers, shattered wine glass,
X-ray of broken bone.

Brittle failure – crack extension

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Snapshots show microscopic processes as a crack extends.

During crack propagation, elastic energy stored in the material is dissipated by breaking atomic bonds

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M.J. Buehler, H. Tang, et al., *Phys. Rev. Letters*, 2007