1.050 Engineering Mechanics I

Lecture 32

Energy bounds in beam structures (cont'd) - How to solve problems

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1.050 – Content overview

I. Dimensional analysis
1. On monsters, mice and mushrooms
2. Similarity relations: Important engineering tools

Lectures 1-3
Sept.

II. Stresses and strength
3. Stresses and equilibrium
4. Strength models (how to design structures, foundations.. against mechanical failure)

Lectures 4-15
Sept./Oct.

III. Deformation and strain
5. How strain gages work?
6. How to measure deformation in a 3D structure/material?

Lectures 16-19
Oct.

IV. Elasticity
7. Elasticity model – link stresses and deformation
8. Variational methods in elasticity

Lectures 20-32
Oct./Nov.

V. How things fail – and how to avoid it
9. Elastic instabilities
10. Plasticity (permanent deformation)
11. Fracture mechanics

Lectures 33-37
Dec.

1.050 – Content overview

I. Dimensional analysis

II. Stresses and strength

III. Deformation and strain

IV. Elasticity
- **…**
- **Lecture 27: Introduction: Energy bounds in linear elasticity (1D system)**
- **Lecture 28: Introduction: Energy bounds in linear elasticity (1D system), cont'd**
- **Lecture 29: 1D examples**
- **Lecture 30: Generalization to 3D**
- **Lecture 31: Energy bounds in beam structures**
- **Lecture 32: Energy bounds in beam structures (cont'd): How to solve problems**

V. How things fail – and how to avoid it
- **Lecture 33 (Mon): Buckling (loss of convexity)**
- **Lecture 34 (Wed): Fracture mechanics I (and surprise!)**
- **Lecture 35 (Fri): Fracture mechanics II**
- **Lecture 36 (Mon): Plastic yield**
- **Lecture 37 (Wed): Wrap-up plastic yield and closure**

Review: 3D isotropic elasticity

$$\boxed{\underset{\underline{\underline{\sigma}}'\,\mathrm{S.A.}}{-\varepsilon_{\mathrm{com}}(\underline{\underline{\sigma}}')} \le \left\{ \begin{matrix} \underset{\underline{\underline{\sigma}}'\,\mathrm{S.A.}}{\max}\left(-\varepsilon_{\mathrm{com}}(\underline{\underline{\sigma}}')\right) \\ \text{is equal to} \\ \underset{\vec{\xi}'\,\mathrm{K.A.}}{\min}\ \varepsilon_{\mathrm{pot}}(\vec{\xi}') \end{matrix} \right\} \le \underset{\vec{\xi}'\,\mathrm{K.A.}}{\varepsilon_{\mathrm{pot}}(\vec{\xi}')}}$$

Lower bound
Complementary energy approach

Solution

Upper bound
Potential energy approach

$$\varepsilon_{\mathrm{com}}(\underline{\underline{\sigma}}') = \psi^{*}(\underline{\underline{\sigma}}') - W^{*}(\vec{T}')$$

$$\varepsilon_{\mathrm{pot}}(\vec{\xi}') = \psi(\underline{\underline{\varepsilon}}') - W(\vec{\xi}')$$

Beam structures (2D)

Complementary free energy

$$\psi^* = \int_{x=0..l} \left[\frac{1}{2}\frac{N^2}{ES} + \frac{1}{2}\frac{M_y^2}{EI} \right] dx$$

Free energy

$$\psi = \int_{x=0..l} \left[\frac{1}{2} ES\left(\varepsilon_{xx}^0\right)^2 + \frac{1}{2} EI\left(\vartheta_y^0\right)^2 \right] dx$$

Note: For 2D, the only contributions are axial forces & moments and axial strains and curvatures (general 3D case see manuscript page 263 and following)

Beam structures

External work by prescribed displacements

$$W^* = \sum_i \left[\vec{\xi}^d(x_i) \cdot \vec{R} + \omega_y^d(x_i) M_{y,R} \right] = \sum_i \left[\xi_x^d(x_i) R_x + \xi_z^d(x_i) R_z + \omega_y^d(x_i) M_{y,R} \right]$$

External work by prescribed force densities/forces/moments

$$W = \int_{x=0..l} \vec{\xi}^0 \cdot \vec{f}^d(x) dx + \sum_i \left[\vec{\xi}^0 \cdot \vec{F}^d(x_i) + \omega_y M_y^d(x_i) \right]$$

$$= \int_{x=0..l} \left[\xi_x^0 f_x^d(x_i) + \xi_z^0 f_z^d(x_i) \right] dx + \sum_i \left[\xi_x^0 F_x^d(x_i) + \xi_z^0 F_z^d(x_i) + \omega_y M_y^d(x_i) \right]$$

Clapeyron’s formulas

$$\boxed{\begin{aligned} \psi &= \psi^* = \frac{1}{2}(W^* + W) \\ \varepsilon_{pot} &= \frac{1}{2}(W^* - W) \\ \varepsilon_{com} &= \frac{1}{2}(W - W^*) \end{aligned}}$$

Significance: Calculate solution potential/complementary energy (“target”) from BCs

Beam elasticity

$$\boxed{-\underset{N',M_y'\text{S.A.}}{\varepsilon_{\text{com}}(\vec{F}_S', M_y')} \leq \left\{ \begin{matrix} \max\limits_{N',M_y'\ \text{S.A.}} \left(-\varepsilon_{\text{com}}(\vec{F}_S', M_y') \right) \\ \text{is equal to} \\ \min\limits_{\vec{\xi}'\text{K.A.}} \varepsilon_{\text{pot}}(\vec{\xi}', \omega_y') \end{matrix} \right\} \leq \underset{\vec{\xi}'\text{K.A.}}{\varepsilon_{\text{pot}}(\vec{\xi}', \omega_y')}}$$

Lower bound	Solution	Upper bound
Complementary energy approach “Stress approach” *Work with unknown but S.A. moments and forces*	$\vec{F}_S', M_y'$ that provide absolute max of $-\varepsilon_{\text{com}}$ $\vec{\xi}', \omega_y'$ that provide absolute min of ε_{pot}	Potential energy approach “Displacement approach) *Work with unknown but K.A. displacements*

Step-by-step solution approach

Use complementary energy approach!

- **Step 1:** Express target solution (Clapeyron's formulas) – calculate complementary energy AT solution
- **Step 2:** Determine reaction forces and reaction moments
- **Step 3:** Determine force and moment distribution, as a function of reaction forces and reaction moments (need M_y and N)
- **Step 4:** Express complementary energy as function of reaction forces and reaction moments (integrate)
- **Step 5:** Minimize complementary energy (take partial derivatives w.r.t. all unknown reaction forces and reaction moments and set to zero); result: set of unknown reaction forces and moments that minimize the complementary energy
- **Step 6:** Calculate complementary energy at the minimum (based on resulting forces and moments obtained in step 5)
- **Step 7:** Make comparison with target solution = find solution displacement

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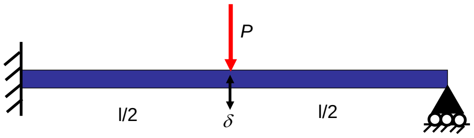
Example



δ = unknown displacement at point of load application

Structure is statically indeterminate to degree 1

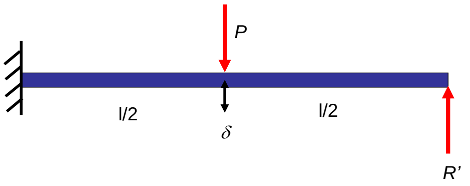
Can not be solved by relying on static equilibrium only (too many unknown forces, 'hyperstatic').

Goal: Solve problem using complementary energy approach

Example

Step 1: Target solution $\varepsilon_{com} = \frac{1}{2} P\delta$

Step 2: Determine hyperstatic forces and moments (here: *R'*)



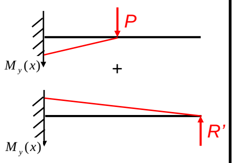
Hyperstatic force

Example

Step 3: Determine force and moment distribution (as a function of hyperstatic force *R')*:

Concept of superposition often helpful

$$M_y(x) = \begin{cases} \frac{Pl}{2}(1-\frac{2x}{l}) - R'l(1-\frac{x}{l}) & 0 \le x \le l/2 \\ -R'l(1-\frac{x}{l}) & l/2 < x \le l \end{cases}$$



Note: Only need expression for *N* and *M_y*

Step 4: Express complementary energy

$$\varepsilon_{com} = \psi^* - W^* = \int_{x=0..l} \left[\overset{=0}{\cancel{\frac{1}{2}\frac{N^2}{ES}}} + \frac{1}{2}\frac{M_y^2}{EI} \right] dx - \overset{=0}{\cancel{W^*}}$$

No contribution from prescribed displacements

$$\varepsilon_{com}(R') = \frac{1}{2EI}\left(\frac{l^3}{3}R'^2 - \frac{5}{24} l^3 R' P + \frac{1}{24} l^3 P^2 \right)$$

Example

Step 5: Find min of $\varepsilon_{com}(R')$

$$\frac{\partial \varepsilon_{com}(R')}{\partial R'} = 0$$

$$\frac{1}{2EI}\left(\frac{2l^3 R'}{3} - \frac{5}{24} l^3 P\right) = 0$$

$$R' = \frac{5}{16} P$$

Step 6: Minimum complementary energy

$$\varepsilon_{com}(R' = \frac{5}{16} P) = \frac{7}{1536EI} l^3 P$$

Example

Step 7: Compare with target solution

$$\varepsilon_{com} = \frac{1}{2} P\delta \le \varepsilon_{com}(R' = \frac{5}{16} P) = \frac{7}{1536EI} l^3 P$$

$$\delta \le \frac{7}{768EI} l^3 P$$

$\delta = \frac{7}{768EI} l^3 P$ represents a minimum of the complementary energy

Is it a global minimum, that is, the solution?

1. M_y' is S.A.
2. *R'* is the only hyperstatic reaction force (in other words, the only source of additional moments)
3. Therefore, the minimum is actually a global minimum, and therefore, it is the solution

Generalization (important)

- For any homogeneous beam problem, the minimization of the complementary energy with respect to all hyperstatic forces and moments

$$X_i = \{R_i, M_{y,R;i}\}$$

yields the **solution of the linear elastic beam problem**:

$$\frac{\partial}{\partial X_i}\left(\varepsilon_{\text{com}}(X_i)\right) \overset{!}{=} 0$$

$$\frac{1}{2}\left(W - W^*\right) \equiv \min_{X_i} \varepsilon_{\text{com}}(X_i)$$