1.050 Engineering Mechanics I

Lecture 29

Energy bounds in 1D systems

Examples and applications

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1.050 – Content overview

I. Dimensional analysis
1. On monsters, mice and mushrooms
2. Similarity relations: Important engineering tools

Lectures 1-3
Sept.

II. Stresses and strength
3. Stresses and equilibrium
4. Strength models (how to design structures, foundations.. against mechanical failure)

Lectures 4-15
Sept./Oct.

III. Deformation and strain
5. How strain gages work?
6. How to measure deformation in a 3D structure/material?

Lectures 16-19
Oct.

IV. Elasticity
7. Elasticity model – link stresses and deformation
8. Variational methods in elasticity

Lectures 20-31
Oct./Nov.

V. How things fail – and how to avoid it
9. Elastic instabilities
10. Plasticity (permanent deformation)
11. Fracture mechanics

Lectures 32-37
Dec.

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1.050 – Content overview

I. Dimensional analysis

II. Stresses and strength

III. Deformation and strain

IV. Elasticity

…
Lecture 23: Applications and examples
Lecture 24: Beam elasticity
Lecture 25: Applications and examples (beam elasticity)
Lecture 26: … cont'd and closure
Lecture 27: Introduction: Energy bounds in linear elasticity (1D system)
Lecture 28: Introduction: Energy bounds in linear elasticity (1D system), cont'd
Lecture 29: 1D examples
Lecture 30: Generalization to 3D
…

V. How things fail – and how to avoid it
Lectures 32 to 37

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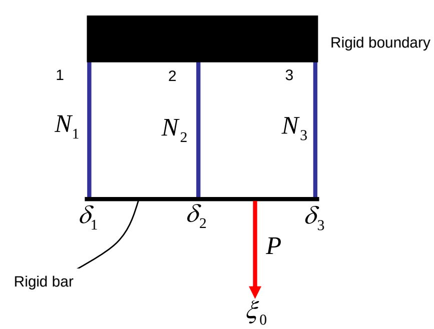
Example system: 1D truss structure



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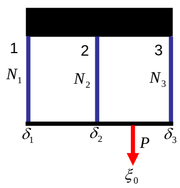
Minimum potential energy approach

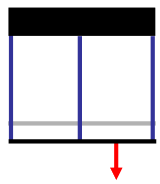
Conditions for kinematically admissible (K.A.): Deformation must be compatible w/ rigid bar



Prescribed force
Unknown displacement

Consider two kinematically admissible (K.A.) displacement fields

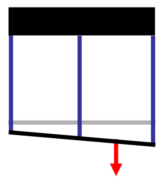
Approximation to solution (K.A.)



(1)

$\delta_1' = \delta_2' = \delta_3' = \xi_0'$

Actual solution



(2)

δ_1

$\delta_2 = \delta_1 + \frac{2}{3}(\xi_0 - \delta_1)$

$\delta_3 = \delta_1 + \frac{4}{3}(\xi_0 - \delta_1)$

Minimum potential energy approach

$$\varepsilon_{\text{pot}}(\delta_i, \xi_0) = \psi(\delta_i) - P\xi_0 \le \psi(\delta_i') - P\xi_0' = \varepsilon_{\text{pot}}(\delta_i', \xi_0')$$

Potential energy of actual solution is always smaller than the solution to any other displacement field

Therefore, the actual solution realizes a minimum of the potential energy:

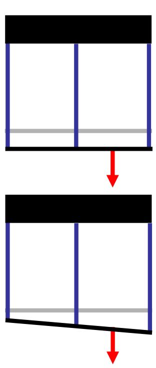
$$\varepsilon_{\text{pot}}(\delta_i, \xi_i) = \min_{\delta_i' \text{K.A.}} \varepsilon_{\text{pot}}(\delta_i', \xi_i')$$

To find a solution, minimize the potential energy for a selected choice of kinematically admissible displacement fields

We have not invoked the EQ conditions!

Minimum potential energy approach

Approximation to solution (K.A.)



(1)

$\delta_1' = \delta_2' = \delta_3' = \xi_0'$

$$\varepsilon_{\text{pot}}(\xi_0') = -\frac{1}{6k}P^2$$

$>$ is larger than

Actual solution

(2)

δ_1

$\delta_2 = \delta_1 + \frac{2}{3}(\xi_0 - \delta_1)$

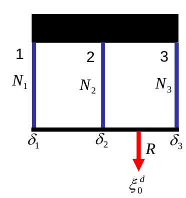
$\delta_3 = \delta_1 + \frac{4}{3}(\xi_0 - \delta_1)$

$$\varepsilon_{\text{pot}}(\xi_0, \delta_1) = -\frac{11}{48k}P^2$$

Minimum complementary energy approach

Conditions for statically admissible (S.A.)

$$\begin{cases} N_1 + N_2 + N_3 = R \\ 3N_1 + N_2 - N_3 = 0 \end{cases}$$

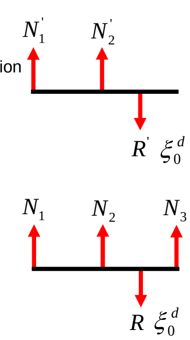


Prescribed displacement
Unknown force

Consider two statically admissible force fields

Approximation to solution Still S.A.

Actual solution (obtained in lecture 20)



(1)

N_1', N_2'

(2)

$N_1 = 1/12R$

$N_2 = 1/3R$

$N_3 = 7/12R$

Minimum complementary energy approach

$$\varepsilon_{\text{com}}(N_i, R) = \psi^*(N_i) - \xi_0^d R \le \psi^*(N_i^{'}) - \xi_0^d R^{'} = \varepsilon_{\text{com}}(N_i^{'}, R^{'})$$

Complementary energy of actual solution is always smaller than the solution to any other displacement field

Therefore, the actual solution realizes a minimum of the complementary energy:

$$\varepsilon_{\text{com}}(N_i, R) = \min_{N_i^{'} \text{S.A.}} \varepsilon_{\text{com}}(N_i^{'}, R^{'})$$

To find a solution, minimize the complementary energy for a selected choice of statically admissible force fields

We have not invoked the kinematics of the problem!

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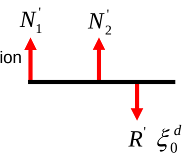
Minimum complementary energy approach

(1)

Approximation to solution Still S.A.



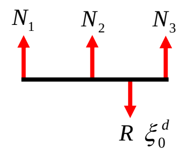
$N_1^{'}, N_2^{'}$

$$\varepsilon_{\text{com}}(R^{'}) = -\frac{1}{5}k\left(\xi_0^d\right)^2$$

$>$ is larger than

(2)

Actual solution (obtained in lecture 20)



$N_1 = 1/12R$

$N_2 = 1/3R$

$N_3 = 7/12R$

$$\varepsilon_{\text{com}}(R, N_1) = -\frac{12}{11}k\left(\xi_0^d\right)^2$$

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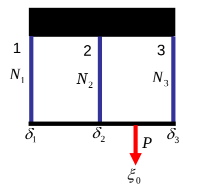
Combine: Upper/lower bound



Prescribed force
Unknown displacement

Treat this BC with complementary energy approach

$$\varepsilon_{\text{com}}(N_1) = \psi^*(N_i) - \xi_0^d R$$

= 0

$$\varepsilon_{\text{com}}(N_1) = \frac{1}{4k}\left(12N_1^2 - 2PN_1 + P^2\right)$$

Find min: $\dfrac{\partial \varepsilon_{\text{com}}(N_1)}{\partial N_1} = 0 \quad N_1 = \dfrac{1}{12}P \longrightarrow \varepsilon_{\text{com}} = \dfrac{11}{48k}P^2$

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Combine: Upper/lower bound

$$-\varepsilon_{\text{com}}(N_i^{'}, R^{'}) \le \left\{ \begin{array}{c} \max\limits_{N_i^{'} \text{S.A.}}\left(-\varepsilon_{\text{com}}(N_i^{'}, R^{'})\right) \\ \text{is equal to} \\ \min\limits_{\delta_i^{'} \text{K.A.}} \varepsilon_{\text{pot}}(\delta_i^{'}, \xi_i^{'}) \end{array} \right\} \le \varepsilon_{\text{pot}}(\delta_i^{'}, \xi_i^{'})$$

Lower bound — Upper bound

$$\varepsilon_{\text{com}} = \frac{11}{48k}P^2 \longrightarrow -\varepsilon_{\text{com}} = -\frac{11}{48k}P^2$$

$$\varepsilon_{\text{pot}}(\xi_0, \delta_1) = -\frac{11}{48k}P^2$$

At the solution to the elasticity problem, the upper and lower bound coincide

Another example

Approximate solution for coupled beam-truss problem

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Step-by-step approach

- **Step 1:** Determine K.A. displacement field (for approximation, find appropriate assumed displacement field)
- **Step 2:** Express work balance – find $\varepsilon_{\text{pot}} / \varepsilon_{\text{com}}$
- **Step 3:** Find min of $\varepsilon_{\text{pot}} / \varepsilon_{\text{com}}$
- **Step 4:** Determine displacement field, forces etc.
- **Solution is <u>approximation</u> to actual solution**

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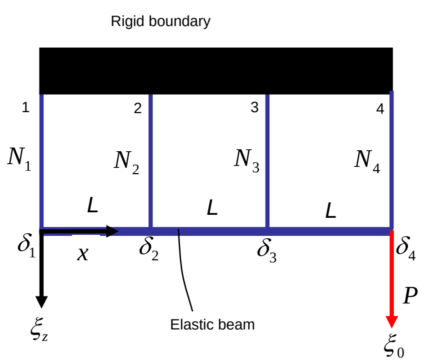

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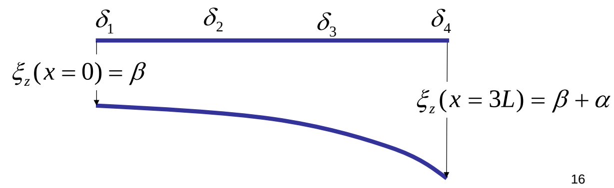
Minimum potential energy approach

Step 1: Assume K.A. displacement field

$$\xi_z(x;\alpha,\beta) = \beta + \alpha\left(\frac{x}{3L}\right)^2$$

(approximation of the actual solution…)



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Minimum potential energy approach

Displacement of the four truss members

(*)
$$\begin{cases} \delta_1 = \xi_z(x=0) = \beta \\ \delta_2 = \xi_z(x=L) = \beta + \frac{\alpha}{9} \\ \delta_3 = \xi_z(x=2L) = \beta + \frac{4}{9}\alpha \\ \delta_4 = \xi_z(x=3L) = \beta + \alpha \\ \xi_0 = \delta_4 \end{cases}$$

$$\xi_z(x;\alpha,\beta) = \beta + \alpha\left(\frac{x}{3L}\right)^2$$

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Minimum potential energy approach

Step 2:
Total free energy of a beam:

Page 215 in manuscript (chapter 5)

$$\psi_B = \int_{x=0}^{3L}\int_{z=-h/2}^{h/2}\int_{y=-b/2}^{b/2} \frac{1}{2}E\left(\varepsilon_{xx}^0 + \vartheta_y^0 z\right)^2 dydzdx$$

with: $\varepsilon_{xx}^0 = 0$ (no displacement in the *x*-direction)

$$\vartheta_y^0 = -\frac{\partial^2 \xi_z}{\partial x^2} = -\frac{2\alpha}{9L^2}$$ (curvature can be calculated from the assumed displacement field)

$$\psi_B = \frac{E}{2}\frac{4\alpha^2}{81L^4}\int_{x=0}^{3L}\int_{z=-h/2}^{h/2}\int_{y=-b/2}^{b/2} z^2 dydzdx \qquad \psi_B(\alpha,\beta) = \underbrace{\frac{bh^3E}{162L^3}}_{=:\frac{1}{2}k_B}\alpha^2$$

"spring constant"

Minimum potential energy approach

$$\psi_B(\alpha) = \frac{1}{2}k_B\alpha^2$$

Total free energy:

Sum of free energies of four trusses… $\psi_i = \frac{1}{2}k\delta_i^2$

$$\psi(\alpha,\beta) = \psi_B(\alpha) + \sum_{i=1..4}\psi_i(\alpha,\beta)$$

$$\psi(\alpha,\beta) = \frac{1}{2}k_B\alpha^2 + \frac{1}{2}k\left(\beta^2 + \left(\beta+\frac{\alpha}{9}\right)^2 + \left(\beta+\frac{4\alpha}{9}\right)^2 + (\beta+\alpha)^2\right)$$

External work $W = F(\alpha+\beta)$

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Minimum potential energy approach

$$\varepsilon_{\text{pot}}(\alpha,\beta) = \frac{1}{2}k_B\alpha^2 + \frac{1}{2}k\left(\beta^2 + \left(\beta+\frac{\alpha}{9}\right)^2 + \left(\beta+\frac{4\alpha}{9}\right)^2 + (\beta+\alpha)^2\right) - F(\beta+\alpha)$$

Step 3: $\min_{\alpha,\beta}\left(\varepsilon_{\text{pot}}(\alpha,\beta)\right)$

How to find minimum of this function?

Take partial derivatives, and set each to zero

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Minimum potential energy approach

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{2}k_B\alpha^2+\frac{1}{2}k\left(\beta^2+\left(\beta+\frac{\alpha}{9}\right)^2+\left(\beta+\frac{4\alpha}{9}\right)^2+(\beta+\alpha)^2\right)-F(\beta+\alpha)\right)=0$$

$$\frac{\partial}{\partial\beta}\left(\frac{1}{2}k_B\alpha^2+\frac{1}{2}k\left(\beta^2+\left(\beta+\frac{\alpha}{9}\right)^2+\left(\beta+\frac{4\alpha}{9}\right)^2+(\beta+\alpha)^2\right)-F(\beta+\alpha)\right)=0$$

Results in a system of linear equations:

$$\begin{pmatrix} k_B+\frac{98}{81}k & \frac{14}{9}k \\ \frac{14}{9}k & 4k \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}F\\ F\end{pmatrix}$$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix} k_B+\frac{98}{81}k & \frac{14}{9}k \\ \frac{14}{9}k & 4k \end{pmatrix}^{-1}\begin{pmatrix}F\\ F\end{pmatrix}$$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}\dfrac{99F}{2(81k_B+49k)}\\ \dfrac{F(81k_B-28k)}{4k(81k_B+49k)}\end{pmatrix}$$

Step 4: Based on solution, determine displacement field δ_i (from (*)), then forces: $N_i = k\delta_i$

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