1.050 Engineering Mechanics I

Lecture 28

Introduction: Energy bounds in linear elasticity (cont'd)

1.050 – Content overview

I. Dimensional analysis
1. On monsters, mice and mushrooms
2. Similarity relations: Important engineering tools

Lectures 1-3
Sept.

II. Stresses and strength
3. Stresses and equilibrium
4. Strength models (how to design structures, foundations.. against mechanical failure)

Lectures 4-15
Sept./Oct.

III. Deformation and strain
5. How strain gages work?
6. How to measure deformation in a 3D structure/material?

Lectures 16-19
Oct.

IV. Elasticity
7. Elasticity model – link stresses and deformation
8. Variational methods in elasticity

Lectures 20-31
Oct./Nov.

V. How things fail – and how to avoid it
9. Elastic instabilities
10. Plasticity (permanent deformation)
11. Fracture mechanics

Lectures 32-37
Dec.

1.050 – Content overview

I. Dimensional analysis

II. Stresses and strength

III. Deformation and strain

IV. Elasticity

…
Lecture 23: Applications and examples
Lecture 24: Beam elasticity
Lecture 25: Applications and examples (beam elasticity)
Lecture 26: … cont'd and closure
Lecture 27: Introduction: Energy bounds in linear elasticity (1D system)
Lecture 28: Introduction: Energy bounds in linear elasticity (1D system), cont'd
Lecture 29: Generalization to 3D, examples
…

V. How things fail – and how to avoid it
Lectures 32 to 37

Outline and goals

Use concept of concept of convexity to derive conditions that specify the solutions to elasticity problems

Obtain two approaches:

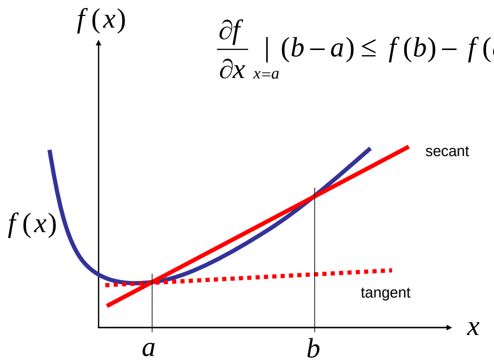
Approach 1: Based on minimizing the potential energy

Approach 2: Based on minimizing the complementary energy

Last part: Combine the two approaches: **Upper/lower bound**

Reminder: convexity of a function

$$\frac{\partial f}{\partial x}\Big|_{x=a}(b-a) \leq f(b) - f(a)$$



Free energy and complementary free energy functions ψ_i^*, ψ_i are convex!

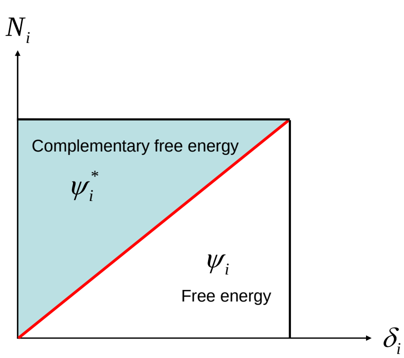
Total external work

$$W^d = \vec{\xi} \cdot \vec{F}^d + \vec{\xi}^d \cdot \vec{R}$$

Work done by prescribed forces Displacement s unknown

Work done by prescribed displacements, force unknown

Total internal work



$$\sum_i \delta_i N_i = \psi^*(N_i) + \psi(\delta_i)$$

State equations

$$N_i = \frac{\partial \psi}{\partial \delta_i}$$

$$\delta_i = \frac{\partial \psi^*}{\partial N_i}$$

Combining it…

$$W^d = \vec{\xi} \cdot \vec{F}^d + \vec{\xi}^d \cdot \vec{R} \overset{!}{=} \psi + \psi^*$$

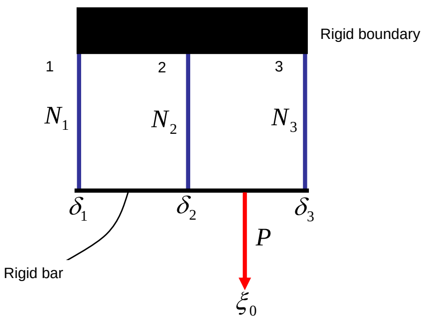
$$-\left(\psi^* - \vec{\xi}^d \cdot \vec{R}\right) \overset{!}{=} \psi - \vec{\xi} \cdot \vec{F}^d$$

Complementary energy $=: \varepsilon_{\text{com}}$

Potential energy $=: \varepsilon_{\text{pot}}$

Solution to elasticity problem $-\varepsilon_{\text{com}} = \varepsilon_{\text{pot}}$

Example system: 1D truss structure



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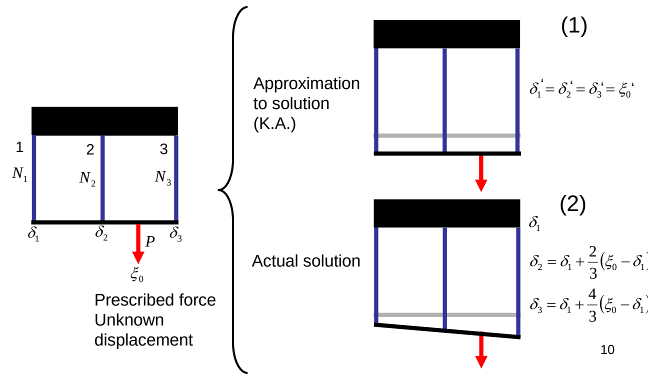
Minimum potential energy approach

Consider two kinematically admissible (K.A.) displacement fields



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Minimum potential energy approach

(1) $\xi_0' P = \sum N_i \delta_i'$

(2) $\xi_0 P = \sum N_i \delta_i$

(1)-(2) $P(\xi_0' - \xi_0) = \sum_i N_i (\delta_i' - \delta_i) = \sum_i \frac{\partial \psi}{\partial \delta_i} (\delta_i' - \delta_i)$

$$N_i = \frac{\partial \psi}{\partial \delta_i}$$

Convexity: $\frac{\partial \psi}{\partial \delta_i} (\delta_i' - \delta_i) \le \psi(\delta_i') - \psi(\delta_i)$

$$P(\xi_0' - \xi_0) \le \psi(\delta_i') - \psi(\delta_i)$$

$$\varepsilon_{\text{pot}}(\delta_i, \xi_0) = \psi(\delta_i) - P\xi_0 \le \psi(\delta_i') - P\xi_0' = \varepsilon_{\text{pot}}(\delta_i', \xi_0')$$

Minimum potential energy approach

$$\varepsilon_{\text{pot}}(\delta_i, \xi_0) = \psi(\delta_i) - P\xi_0 \le \psi(\delta_i') - P\xi_0' = \varepsilon_{\text{pot}}(\delta_i', \xi_0')$$

Potential energy of actual solution is always smaller than the solution to any other displacement field

Therefore, the actual solution realizes a minimum of the potential energy:

$$\varepsilon_{\text{pot}}(\delta_i, \xi_i) = \min_{\delta_i' \text{K.A.}} \varepsilon_{\text{pot}}(\delta_i', \xi_i')$$

To find a solution, minimize the potential energy for a selected choice of kinematically admissible displacement fields

We have not invoked the EQ conditions!

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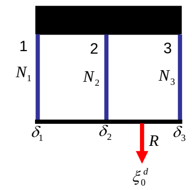
Minimum complementary energy approach

Conditions for statically admissible (S.A.)

$$\begin{cases} N_1 + N_2 + N_3 = R \\ 3N_1 + N_2 - N_3 = 0 \end{cases}$$

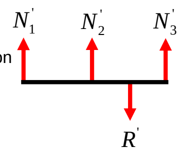


Prescribed displacement
Unknown force

Consider two statically admissible force fields

(1)

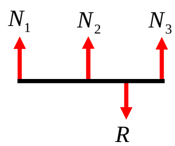
Approximation to solution
Still S.A.



$N_1^{'}, N_2^{'}, N_3^{'}$

(2)

Actual solution (obtained in lecture 20)



$N_1 = 1/12R$

$N_2 = 1/3R$

$N_3 = 7/12R$

Minimum complementary energy approach

(1) $\xi_0^d R^{'} = \sum N_i^{'} \delta_i$

(2) $\xi_0^d R = \sum N_i \delta_i$

(1)-(2) $$\xi_0^d (R^{'} - R) = \sum_i \delta_i (N_i^{'} - N_i) = \sum_i \frac{\partial \psi^*(N_i)}{\partial N_i}(N_i^{'} - N_i)$$

$$\delta_i = \frac{\partial \psi^*}{\partial N_i}$$

Convexity: $$\frac{\partial \psi^*}{\partial N_i}(N_i^{'} - N_i) \le \psi^*(N_i^{'}) - \psi^*(N_i)$$

$$\xi_0^d (R^{'} - R) \le \psi^*(N_i^{'}) - \psi^*(N_i)$$

$$\varepsilon_{\text{com}}(N_i, R) = \psi^*(N_i) - \xi_0^d R \le \psi^*(N_i^{'}) - \xi_0^d R^{'} = \varepsilon_{\text{com}}(N_i^{'}, R^{'})$$

Minimum complementary energy approach

$$\varepsilon_{\text{com}}(N_i, R) = \psi^*(N_i) - \xi_0^d R \le \psi^*(N_i^{'}) - \xi_0^d R^{'} = \varepsilon_{\text{com}}(N_i^{'}, R^{'})$$

Complementary energy of actual solution is always smaller than the solution to any other displacement field

Therefore, the actual solution realizes a minimum of the complementary energy:

$$\varepsilon_{\text{com}}(N_i, R) = \min_{N_i^{'} \text{S.A.}} \varepsilon_{\text{com}}(N_i^{'}, R^{'})$$

To find a solution, minimize the complementary energy for a selected choice of statically admissible force fields

We have not invoked the kinematics of the problem!

Combine: Upper/lower bound

Recall that the solution to elasticity problem $-\varepsilon_{\text{com}} = \varepsilon_{\text{pot}}$

Therefore

$$-\varepsilon_{\text{com}}(N_i, R) = \max_{N_i^{'} \text{S.A.}}\left(-\varepsilon_{\text{com}}(N_i^{'}, R^{'})\right) \quad \text{(change sign)}$$

$$\varepsilon_{\text{pot}}(\delta_i, \xi_i) = \min_{\delta_i^{'} \text{K.A.}} \varepsilon_{\text{pot}}(\delta_i^{'}, \xi_i^{'})$$

$$-\varepsilon_{\text{com}}(N_i^{'}, R^{'}) \le \left\{ \begin{array}{c} \max_{N_i^{'} \text{S.A.}}\left(-\varepsilon_{\text{com}}(N_i^{'}, R^{'})\right) \\ \text{is equal to} \\ \min_{\delta_i^{'} \text{K.A.}} \varepsilon_{\text{pot}}(\delta_i^{'}, \xi_i^{'}) \end{array} \right\} \le \varepsilon_{\text{pot}}(\delta_i^{'}, \xi_i^{'})$$

Lower bound | Upper bound

At the solution to the elasticity problem, the upper and lower bound coincide

Approach to approximate/numerical solution of elasticity problems

- **Minimum potential energy approach:**
Select a guess for a displacement field; the only condition that must be satisfied is that it is kinematically admissible. In a numerical solution, this displacement field is typically a function of some unknown parameters ($a_1,a_2,\ldots$)
  - Express the potential energy as a function of the unknown parameters $a_1,a_2,\ldots$
  - Minimize the potential energy by finding the appropriate set of parameters ($a_1,a_2,\ldots$) for the minimum – generally yields approximate solution
  - The actual solution is given by the displacement field that yields a total minimum of the potential energy. Otherwise, an approximate solution is obtained
- **Minimum complementary energy approach:**
Select a guess for a force field; the only condition that must be satisfied is that it is statically admissible. In a numerical solution, this force field is typically a function of some unknown parameters ($b_1,b_2,\ldots$)
  - Express the complementary energy as a function of the unknown parameters $b_1,b_2,\ldots$
  - Minimize the complementary energy by finding the appropriate set of parameters ($b_1,b_2,\ldots$) for the minimum – generally yields approximate solution
  - The actual solution is given by the force field that yields a total minimum of the complementary energy. Otherwise, an approximate solution is obtained
- **At the elastic solution, the minimum potential energy approach solution and the negative of the solution of the minimum complementary energy approach coincide**