1.050 Engineering Mechanics

Lecture 24:
Beam elasticity – derivation of governing equation

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1.050 – Content overview

I. Dimensional analysis

1. On monsters, mice and mushrooms
2. Similarity relations: Important engineering tools

Lectures 1-3
Sept.

II. Stresses and strength

3. Stresses and equilibrium
4. Strength models (how to design structures, foundations.. against mechanical failure)

Lectures 4-15
Sept./Oct.

III. Deformation and strain

5. How strain gages work?
6. How to measure deformation in a 3D structure/material?

Lectures 16-19
Oct.

IV. Elasticity

7. Elasticity model – link stresses and deformation
8. Variational methods in elasticity

Lectures 20-31
Oct./Nov.

V. How things fail – and how to avoid it

9. Elastic instabilities
10. Plasticity (permanent deformation)
11. Fracture mechanics

Lectures 32-37
Dec.

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1.050 – Content overview

I. Dimensional analysis

II. Stresses and strength

III. Deformation and strain

IV. Elasticity

Lecture 20: Introduction to elasticity (thermodynamics)
Lecture 21: Generalization to 3D continuum elasticity
Lecture 22: Special case: isotropic elasticity
Lecture 23: Applications and examples
Lecture 24: Beam elasticity
Lecture 25: Applications and examples (beam elasticity)
Lecture 26: … cont'd and closure
…

V. How things fail – and how to avoid it

Goal of this lecture

- Derive differential equations that can be solved to determine stress, strain and displacement fields in beam
- Consider 2D beam geometry:

+ boundary conditions (force, clamped, moments…)

- **Approach:** Utilize beam stress model, strain model for beams and combine with isotropic elasticity

Stress

$$(\sigma_{ij}) = \begin{pmatrix} \sigma_{xx} & 0 & \sigma_{xz} \\ 0 & 0 & 0 \\ \sigma_{xz} & 0 & 0 \end{pmatrix}$$

Shape of stress tensor for 2D beam problem

$$N = \int_S \sigma_{xx} dS \quad Q_z = \int_S \sigma_{xz} dS$$

$$M_y = \int_S z\sigma_{xx} dS$$

$$\frac{dM_y}{dx} = Q_z \quad \frac{d^2 M_y}{dx^2} = -f_z$$

$$\frac{dN}{dx} = -f_x$$

Strain

Navier-Bernouilli beam model

$$\varepsilon_{xx} = \varepsilon_{xx}^0 + \vartheta_y^0 z$$

$$\vartheta_y^0 = -\frac{d^2 \xi_z^0}{dx^2} \quad \text{Curvature}$$

$$\varepsilon_{xx}^0 = \frac{d\xi_x^0}{dx} \quad \text{Axial strain}$$

Thus:

$$\varepsilon_{xx} = \frac{d\xi_x^0}{dx} - \frac{d^2 \xi_z^0}{dx^2} z$$

Strain completely determined from displacement of beam reference axis

Isotropic elasticity: $\underline{\underline{\sigma}} = \left(K - \frac{2}{3} G \right) \varepsilon_v \underline{\underline{1}} + 2G \underline{\underline{\varepsilon}}$

Derivation of beam constitutive equation in 3-step approach

Section number below corresponds to section numbering used in class

Step 1: Consider continuum scale alone (derive a relation between stress and strain for the particular shape of the stress tensor in beam geometry)
2.1)

Step 2: Link continuum scale with section scale (use reduction formulas)
2.2)

Step 3: Link section scale to structural scale (beam EQ equations)
2.3)

Overview

Continuum scale

$\underline{\underline{\sigma}}, \underline{\underline{\varepsilon}}$

Section scale

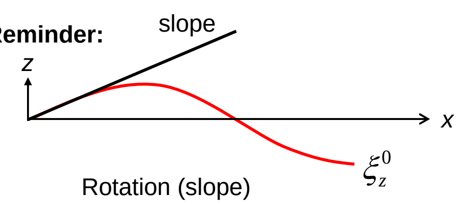
N, Q_z, M_y

Structural scale

$N(x), Q_z(x), M_y(x),$

$\omega_y(x), \xi_x(x), \xi_z(x)$

Reminder:



Rotation (slope)

$$\omega_y^0 = -\frac{d\xi_z^0}{dx}$$

Curvature (=first derivative of rotation)

$$\vartheta_y^0 = -\frac{d^2\xi_z^0}{dx^2} = \frac{d\omega_y^0}{dx}$$

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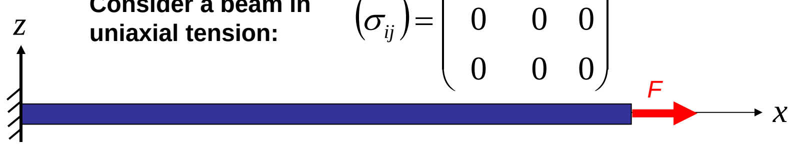
2.1) Step 1 (continuum scale)

Consider a beam in uniaxial tension:

$$(\sigma_{ij}) = \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$$\sigma_{xx} = \left(K - \frac{2}{3}G\right)(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2G\varepsilon_{xx} \quad (1)$$

3 unknowns, 2 equations; can eliminate one variable and obtain relation between 2 remaining ones

$$\begin{cases} \sigma_{yy} = \left(K - \frac{2}{3}G\right)(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2G\varepsilon_{yy} \overset{!}{=} 0 & (2) \\ \sigma_{zz} = \left(K - \frac{2}{3}G\right)(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2G\varepsilon_{zz} \overset{!}{=} 0 & (3) \end{cases}$$

Eqns. (2) and (3) provide relation between ε_{xx} and $\varepsilon_{yy}, \varepsilon_{zz}$:

$$\varepsilon_{yy} = \varepsilon_{zz} = -\underbrace{\frac{1}{2}\frac{3K-2G}{3K+G}}_{=:\nu}\varepsilon_{xx} = -\nu\varepsilon_{xx}$$

Poisson's ratio

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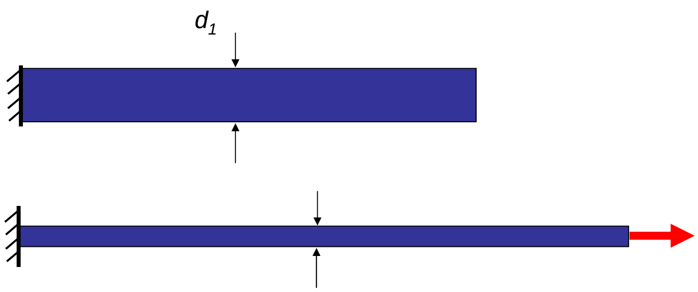
Physical meaning "Poisson's effect"

- The 'Poisson effect' refers to the fact that beams contract in the lateral directions when subjected to tensile strain

$$\boxed{\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx}}$$



$$d_2 = (1+\varepsilon_{yy})d_1 = (1-\nu\varepsilon_{xx})$$

From eq. (1) (with Poisson relation):

$$\sigma_{xx} = \underbrace{\frac{9KG}{3K+G}}_{=:\,E}\varepsilon_{xx}$$

Young's modulus

$$\boxed{\sigma_{xx} = E\varepsilon_{xx}}$$

This result can be generalized: In bending, the shape of the stress tensor is identical, for any point in the cross-section (albeit the component σ_{zz} typically varies with the coordinate *z*)

Thus, the same conditions for the lateral strains applies

Therefore: We can use the same formulas!

$$(\sigma_{ij}) = \begin{pmatrix} \sigma_{xx}(z) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2.2) Step 2 (link to section scale)

Now: Plug in relation $\sigma_{xx} = E\varepsilon_{xx}$ into reduction formulas

Consider that $\varepsilon_{xx} = \frac{d\xi_x^0}{dx} - \frac{d^2\xi_z^0}{dx^2} z$ and thus $\sigma_{xx} = E\left(\frac{d\xi_x^0}{dx} - \frac{d^2\xi_z^0}{dx^2} z\right)$

Results in:

Assume: *E* constant over *S*

$$N = \int_S E\left(\frac{d\xi_x^0}{dx} - \frac{d^2\xi_z^0}{dx^2} z\right) dS \longrightarrow N = E\frac{d\xi_x^0}{dx}\int_S dS - \cancel{E\frac{d^2\xi_z^0}{dx^2}\int_S zdS}^{=0}$$

$$M_y = \int_S E\left(\frac{d\xi_x^0}{dx} z - \frac{d^2\xi_z^0}{dx^2} z^2\right) dS \longrightarrow M_y = \cancel{E\frac{d\xi_x^0}{dx}\int_S zdS}^{=0} - E\frac{d^2\xi_z^0}{dx^2}\underbrace{\int_S z^2 dS}_{=I}$$

Area moment of inertia

$$\boxed{\textbf{Finally:} \quad N = ES\frac{d\xi_x^0}{dx} \qquad M_y = -EI\frac{d^2\xi_z^0}{dx^2}}$$

2.3) Step 4 (link to structural scale)

Beam EQ equations:

$$\frac{d^2M_y}{dx^2} = -f_z$$

$$\frac{dN}{dx} = -f_x$$

with:

$$M_y = -EI\frac{d^2\xi_z^0}{dx^2}$$

$$N = ES\frac{d\xi_x^0}{dx}$$

Beam constitutive equations:

$$M_y = -EI\frac{d^4\xi_z^0}{dx^4} = -f_z$$

$$\boxed{\frac{d^4\xi_z^0}{dx^4} = \frac{f_z}{EI} \qquad \frac{d^2\xi_x^0}{dx^2} = -\frac{f_x}{ES}}$$

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Beam bending elasticity

Governed by this differential equation:

$$\boxed{\frac{d^4 \xi_z^0}{dx^4} = \frac{f_z}{EI}}$$

Integration provides solution for displacement

Solve integration constants by applying BCs

Note:

E = material parameter (Young's modulus)

I = geometry parameter (property of cross-section)

f_z = distributed shear force

How to solve? Lecture 25

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