1.050 Engineering Mechanics

Lecture 23:
Example – detailed steps

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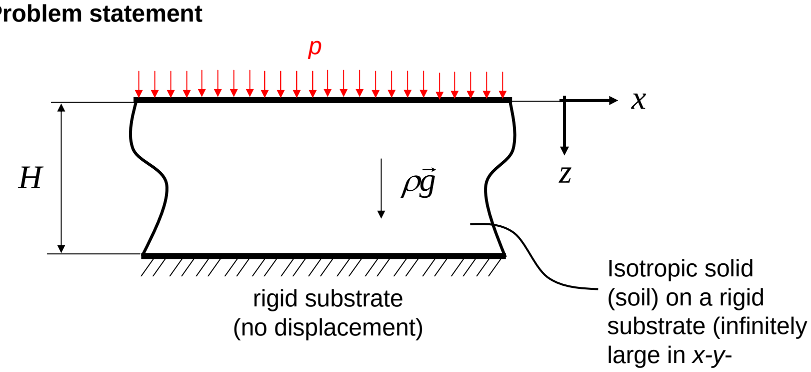
Problem statement



K, G given

Note: *p* is applied pressure at the top of the soil layer

Goal: Determine $\vec{\xi}(\vec{x}), \underline{\underline{\varepsilon}}(\vec{x}), \underline{\underline{\sigma}}(\vec{x})$

On the next few slides we will go through **steps 1, 2, 3 and 4** to solve this problem.

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Reminder: 4-step procedure to solve elasticity problems

- **Step 1**: Write down BCs (stress BCs and displacement BCs), analyze the problem to be solved (read carefully!)
- **Step 2**: Write governing equations for stress tensor, strain tensor, and constitutive equations that link stress and strain, simplify expressions
- **Step 3**: Solve governing equations (e.g. by integration), typically results in expression with unknown integration constants
- **Step 4**: Apply BCs (determine integration constants)

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Step 1: Boundary conditions

Write out all BCs in mathematical equations

Displacement BCs: At z=H: Displacement specified

$\vec{\xi}^d(z=H)=(0,0,0)$ or $\xi_x^d=0, \xi_y^d=0, \xi_z^d=0$

(no displacement at the interface between the soil layer and the rigid substrate)

Stress BCs: At z=0: Stress vector provided $\vec{T}^d(\vec{n}=-\vec{e}_z, z=0)=p\vec{e}_z$

Note: Orientation of surface and C.S.

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Step 2: Governing equations

Write out all governing equations and simplify

Due to the **symmetry of the problem** (infinite in *x*- and *y*-directions), the solution will depend on *z* only, and there are no displacements in the *x*- and *y*-directions (anywhere in the solution domain): $\vec{\xi} = \xi_z \vec{e}_z$

Governing eqn. for strain tensor: $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right)$

Calculation of strain tensor simplifies (symmetry): $\varepsilon_{zz} = \frac{\partial \xi_z}{\partial z}$ (*)

Note : only 1 nonzero coefficient of strain tensor

Governing eqn. for stress tensor: $\text{div}\,\underline{\underline{\sigma}} + \rho\vec{g} = 0$
(cont'd next slide)

Step 2: Governing equations (cont'd)

Governing eqn. for stress tensor:

Gravity only in *z*-direction

$$\cancel{\frac{\sigma_{xx}}{\partial x}} + \cancel{\frac{\sigma_{xy}}{\partial y}} + \frac{\sigma_{xz}}{\partial z} + \cancel{\rho g_x} = 0$$
$$\cancel{\frac{\sigma_{xy}}{\partial x}} + \cancel{\frac{\sigma_{yy}}{\partial y}} + \frac{\sigma_{yz}}{\partial z} + \cancel{\rho g_y} = 0$$
$$\cancel{\frac{\sigma_{xz}}{\partial x}} + \cancel{\frac{\sigma_{yz}}{\partial y}} + \frac{\sigma_{zz}}{\partial z} + \rho g_z = 0$$

Due to symmetry, only dependence on *z*-direction

$$\frac{\sigma_{xz}}{\partial z} = 0 \qquad \frac{\sigma_{yz}}{\partial z} = 0$$

(1) $$\frac{\sigma_{zz}}{\partial z} + \rho g = 0$$

Final set of governing eqns. for stress tensor

(note: $g_z = g$)

Step 2: Governing equations (cont'd)

Link between stress and strain

Linear isotropic elasticity (considering that there is only one nonzero coefficient in the strain tensor, ε_{zz}):

$$\sigma_{11} = \left(K - \frac{2}{3}G\right)\varepsilon_{33}$$
$$\sigma_{22} = \left(K - \frac{2}{3}G\right)\varepsilon_{33}$$
$$\sigma_{33} = \left(K + \frac{4}{3}G\right)\varepsilon_{33} \qquad (2)$$

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Step 2: Governing equations (cont'd)

Now combine eqns. (*), (1) and (2):

Substitute (2) in (1): $\frac{\partial \varepsilon_{zz}}{\partial z}\left(K + \frac{4}{3}G\right) + \rho g = 0 \qquad (4)$

Substitute (*) in (4): $\frac{\partial^2 \xi_z}{\partial z^2}\left(K + \frac{4}{3}G\right) + \rho g = 0$

$$\frac{\partial^2 \xi_z}{\partial z^2} = -\frac{\rho g}{K + \frac{4}{3}G} \qquad (5)$$

$$\frac{\sigma_{xz}}{\partial z} = 0 \quad \frac{\sigma_{yz}}{\partial z} = 0 \qquad (6)$$

Step 2 results in a set of differential eqns.

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Step 3: Solve governing eqns. by integration

From (5):

$$\frac{\partial \xi_z}{\partial z} = -\frac{\rho g}{K+\frac{4}{3}G} z + C_1 = \varepsilon_{zz}$$ (first integration)

$$\sigma_{zz} = \left(K+\frac{4}{3}G\right)\left(-\frac{\rho g}{K+\frac{4}{3}G} z + C_1\right)$$ (knowledge of strain enables to calculate stress via eq. (2))

$$\xi_z = -\frac{1}{2}\frac{\rho g}{K+\frac{4}{3}G} z^2 + C_1 z + C_2$$ (second integration)

From (6):

$$\frac{\sigma_{xz}}{\partial z} = 0 \quad \frac{\sigma_{yz}}{\partial z} = 0 \longrightarrow \sigma_{xz} = \text{const.} = C_3 \quad \sigma_{yz} = \text{const.} = C_4$$

Solution expressed in terms of integration constants C_i

Step 4: Apply BCs

Stress boundary conditions: Integration provided that

$$\sigma_{xz} = \text{const.} = C_3 \quad \sigma_{yz} = \text{const.} = C_4$$

Stress vector at the boundary of the domain:

$$\underbrace{\vec{T}^d(\vec{n} = -\vec{e}_z, z=0) = p\vec{e}_z}_{\text{BC}} \overset{!}{=} \underbrace{-\sigma_{xz}\vec{e}_x - \sigma_{yz}\vec{e}_y - \sigma_{zz}\vec{e}_z}_{\text{Stress vector due to stress tensor at } z=0}$$

Stress vector due to stress tensor at z = 0:

$$\vec{T}(\vec{n} = -\vec{e}_z, z=0) = \underline{\underline{\sigma}}(z=0)\cdot(-\vec{e}_z)$$

Note: Orientation of surface and C.S.

Left and right side must be equal, therefore:

$$\sigma_{xz} = C_3 = 0, \sigma_{yz} = C_4 = 0$$

$$\sigma_{zz} = -p$$

Step 4: Apply BCs (cont'd)

Further,

$$\sigma_{zz} = K + \frac{4}{3}G\left(-\frac{\rho g}{K + \frac{4}{3}G} z + C_1 \right)$$ (general solution)

$$\sigma_{zz}(z=0) = C_1\left(K + \frac{4}{3}G \right) \overset{!}{=} -p$$ (at z=0, see previous slide)

This enables us to determine the constant C_1

$$C_1 = -\frac{p}{K + \frac{4}{3}G}$$

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Step 4: Apply BCs (cont'd)

Displacement boundary conditions:

$$\xi_z = -\frac{1}{2}\frac{\rho g}{K + \frac{4}{3}G} z^2 - \frac{p}{K + \frac{4}{3}G} z + C_2$$ (general solution, with C_1 included)

Displacement is known at $z = H$:

$$\xi_z(z=H) = -\frac{1}{2}\frac{\rho g}{K + \frac{4}{3}G} H^2 - \frac{p}{K + \frac{4}{3}G} H + C_2 \overset{!}{=} 0$$

This enables us to determine the constant C_2

$$C_2 = \frac{1}{K + \frac{4}{3}G}\left(\frac{\rho g}{2} H^2 + pH \right)$$

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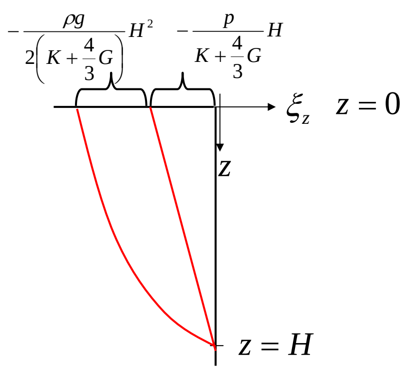
Final solution (summary): Displacement field, strain field, stress field

$$\begin{cases} \xi_z(z) = \dfrac{1}{K+\frac{4}{3}G}\left(\dfrac{\rho g}{2}\left(H^2 - z^2\right) - p(z-H)\right) \\ \varepsilon_{zz}(z) = \dfrac{-\rho g z + p}{K+\frac{4}{3}G} \\ \sigma_{zz}(z) = -\rho g z + p \end{cases}$$

Solution sketch

Displacement profile



Stress profile: