1.050 Engineering Mechanics

Lecture 22:
Isotropic elasticity

1.050 – Content overview

I. Dimensional analysis

1. On monsters, mice and mushrooms
2. Similarity relations: Important engineering tools

Lectures 1-3
Sept.

II. Stresses and strength

3. Stresses and equilibrium
4. Strength models (how to design structures, foundations.. against mechanical failure)

Lectures 4-15
Sept./Oct.

III. Deformation and strain

5. How strain gages work?
6. How to measure deformation in a 3D structure/material?

Lectures 16-19
Oct.

IV. Elasticity

7. Elasticity model – link stresses and deformation
8. Variational methods in elasticity

Lectures 20-31
Oct./Nov.

V. How things fail – and how to avoid it

9. Elastic instabilities
10. Plasticity (permanent deformation)
11. Fracture mechanics

Lectures 32-37
Dec.

1.050 – Content overview

I. Dimensional analysis

II. Stresses and strength

III. Deformation and strain

IV. Elasticity

Lecture 20: Introduction to elasticity (thermodynamics)
Lecture 21: Generalization to 3D continuum elasticity
Lecture 22: Special case: isotropic elasticity
Lecture 23: Applications and examples
…

V. How things fail – and how to avoid it

Important concepts: Isotropic elasticity

- **Isotropic elasticity** = elastic properties do not depend on direction
- In terms of the **free energy change**, this means that the change of the free energy does not depend on the direction of deformation
- Rather, it depends on quantities that are **independent on the direction of deformation** (i.e., independent of coordinate system)
- **Idea:** Use invariants of strain tensor to calculate free energy change
  - Volume change
  - Shape change (shear deformation)
- **Note:** Invariants are defined as properties of strain tensor that are independent of coordinate system (C.S.)

Important mathematical tools

$$\mathrm{tr}\left(\underline{\underline{\varepsilon}}\right) = \underline{\underline{\varepsilon}} : \underline{\underline{1}} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{d\Omega_d - d\Omega_0}{d\Omega_0}$$

Trace of a tensor
Relates to the chain of volume of REV
Independent of C.S. – **trace of a tensor is an invariant**

$$\left|\underline{\underline{\varepsilon}}\right| = \sqrt{\frac{1}{2}\left(\underline{\underline{\varepsilon}} : \underline{\underline{\varepsilon}}^T\right)} = \sqrt{\frac{1}{2}\sum_i \sum_j \varepsilon_{ij}^2}$$

‘Magnitude’ of a tensor (2^{nd} order norm)

Note: Analogy to the ‘magnitude’ of a tensor is the norm of a first order tensor (=vector), that is, its length

Overview: Approach

Step 1: Calculate change in volume $\varepsilon_v = \mathrm{tr}\left(\underline{\underline{\varepsilon}}\right) = \underline{\underline{\varepsilon}} : \underline{\underline{1}}$

Step 2: Calculate magnitude of angle change

Define strain deviator tensor = tensor that describes deformation without the volume change (**trace of strain deviator tensor is zero!**)

$$\underline{\underline{e}} = \left(\underline{\underline{\varepsilon}} - \frac{1}{3}\mathrm{tr}\left(\underline{\underline{\varepsilon}}\right)\underline{\underline{1}} \right) \qquad \mathrm{tr}\left(\underline{\underline{e}}\right) = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} - \frac{1}{3}\left(3\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right)\right) = 0$$

$$\varepsilon_d = 2\left|\underline{\underline{e}}\right| = 2\sqrt{\frac{1}{2}\left(\underline{\underline{e}} : \underline{\underline{e}}^T\right)} = 2\sqrt{\frac{1}{2}\sum_i \sum_j e_{ij}^2}$$

Step 3: Define two coefficients to link energy change with deformation (“spring model”):

$$\Psi = \frac{1}{2}K\varepsilon_v^2 + \frac{1}{2}G\varepsilon_d^2$$

Bulk modulus (→ K)

Shear modulus (→ G)

Note

The approach that the free energy under deformation depends only on volume change and overall angle change is not derived from physical principles

Rather, it is an **assumption,** which is made to ‘model’ the behavior of a solid (**modeling is finding a mathematical representation of a physical phenomenon**)

Generally, models must be validated, for instance through experiments

Alternative approach: Calculation of from ‘first principles’ – by explicitly considering the atomistic scale of atomic, molecular etc. interactions

Spring 2008: 1.021J Introduction to Modeling and Simulation (Buehler, Radovitzky, Marzari) – continuum methods, particle methods, quantum mechanics

Stress-strain relation

Total stress tensor = sum of contribution from volume change and contribution from shape change:

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_v + \underline{\underline{\sigma}}_d$$

$$\underline{\underline{\sigma}}_v = \frac{\partial \Psi_v}{\partial \underline{\underline{\varepsilon}}}$$

$$\underline{\underline{\sigma}}_d = \frac{\partial \Psi_d}{\partial \underline{\underline{\varepsilon}}}$$

Next step: Carry out differentiations

Stress-strain relation

Total stress tensor = sum of contribution from volume change and contribution from shape change:

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_v + \underline{\underline{\sigma}}_d \qquad \underline{\underline{\sigma}}_v = \frac{\partial \Psi_v}{\partial \underline{\underline{\varepsilon}}} \qquad \underline{\underline{\sigma}}_d = \frac{\partial \Psi_d}{\partial \underline{\underline{\varepsilon}}}$$

1. Calculation of $\underline{\underline{\sigma}}_v$

$$\Psi_v = \frac{1}{2} K \varepsilon_v^2 \qquad \underline{\underline{\sigma}}_v = \frac{\partial \Psi_v}{\partial \underline{\underline{\varepsilon}}} = \frac{\partial \Psi_v}{\partial \varepsilon_v} : \frac{\partial \varepsilon_v}{\partial \underline{\underline{\varepsilon}}} = K \varepsilon_v \underline{\underline{1}}$$

$$\frac{\partial \Psi_v}{\partial \varepsilon_v} = K \varepsilon_v \qquad \frac{\partial \varepsilon_v}{\partial \underline{\underline{\varepsilon}}} = \frac{\partial (\mathrm{tr}(\underline{\underline{\varepsilon}}))}{\partial \underline{\underline{\varepsilon}}} = \frac{\partial (\underline{\underline{\varepsilon}} : \underline{\underline{1}})}{\partial \underline{\underline{\varepsilon}}} = \underline{\underline{1}}$$

Stress-strain relation

2. Calculation of $\underline{\underline{\sigma}}_d$

$$\Psi_d = \frac{1}{2} G \varepsilon_d^2$$

$$\underline{\underline{\sigma}}_d = \frac{\partial \Psi_d}{\partial \underline{\underline{\varepsilon}}} = \frac{\partial \Psi_d}{\partial \underline{\underline{e}}} : \frac{\partial \underline{\underline{e}}}{\partial \underline{\underline{\varepsilon}}} = 2G\underline{\underline{e}} : \left(\underline{\underline{\underline{\underline{1}}}} - \frac{1}{3} \underline{\underline{1}} \otimes \underline{\underline{1}} \right) = 2G\underline{\underline{e}} - \frac{1}{3} \left(\underline{\underline{e}} : \underline{\underline{1}} \right) \otimes \underline{\underline{1}}$$

$=0$

since:

$$\mathrm{tr}\left(\underline{\underline{e}}\right) = 0$$

$$\frac{\partial \Psi_d}{\partial \underline{\underline{e}}} = \frac{1}{2} G \frac{\partial \left(2 \underline{\underline{e}} : \underline{\underline{e}}^T \right)}{\partial \underline{\underline{e}}} = 2G\underline{\underline{e}}$$

$$\frac{\partial \underline{\underline{e}}}{\partial \underline{\underline{\varepsilon}}} = \underline{\underline{\underline{\underline{1}}}} - \frac{1}{3} \underline{\underline{1}} \otimes \underline{\underline{1}}$$

Note (definition of ε_d **):**

$$\varepsilon_d = 2\left|\underline{\underline{e}}\right| = 2\sqrt{\frac{1}{2}\left(\underline{\underline{e}} : \underline{\underline{e}}^T \right)}$$

Note (definition of $\underline{\underline{e}}$ **):**

tensor product

$$\underline{\underline{e}} = \underline{\underline{\varepsilon}} - \frac{1}{3} \varepsilon_v \underline{\underline{1}} = \underline{\underline{\varepsilon}} - \frac{1}{3} \left(\underline{\underline{\varepsilon}} : \underline{\underline{1}} \right) \otimes \underline{\underline{1}}$$

Complete stress-strain relation

3. Putting it all together: $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_v + \underline{\underline{\sigma}}_d$

$$\underline{\underline{\sigma}} = K\varepsilon_v \underline{\underline{1}} + 2G\underline{\underline{e}} = K\varepsilon_v \underline{\underline{1}} + 2G\left(\underline{\underline{\varepsilon}} - \frac{1}{3}\varepsilon_v \underline{\underline{1}}\right)$$

$$\underline{\underline{e}} = \left(\underline{\underline{\varepsilon}} - \frac{1}{3}\mathrm{tr}\left(\underline{\underline{\varepsilon}}\right)\underline{\underline{1}}\right)$$

Deviatoric part of the strain tensor

$$\underline{\underline{\sigma}} = \left(K - \frac{2}{3}G\right)\varepsilon_v \underline{\underline{1}} + 2G\underline{\underline{\varepsilon}}$$

Reorganized…

Complete stress-strain relation

$$\underline{\underline{\sigma}} = \left(K - \frac{2}{3}G\right)\varepsilon_v \underline{\underline{1}} + 2G\underline{\underline{\varepsilon}} = \left(K - \frac{2}{3}G\right)\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right)\underline{\underline{1}} + 2G\underline{\underline{\varepsilon}}$$

Writing it out in coefficient form:

$$\begin{cases} \sigma_{11} = \left(K - \frac{2}{3}G\right)\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) + 2G\varepsilon_{11} \\ \sigma_{22} = \left(K - \frac{2}{3}G\right)\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) + 2G\varepsilon_{22} \\ \sigma_{33} = \left(K - \frac{2}{3}G\right)\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) + 2G\varepsilon_{33} \\ \sigma_{12} = 2G\varepsilon_{12} \\ \sigma_{23} = 2G\varepsilon_{23} \\ \sigma_{13} = 2G\varepsilon_{13} \end{cases}$$

Complete stress-strain relation

Rewrite by collecting terms multiplying ε_{ii}

$$\sigma_{11} = \left(K + \frac{4}{3}G\right)\varepsilon_{11} + \left(K - \frac{2}{3}G\right)\varepsilon_{22} + \left(K - \frac{2}{3}G\right)\varepsilon_{33} \qquad (1)$$

$$\sigma_{22} = \left(K - \frac{2}{3}G\right)\varepsilon_{11} + \left(K + \frac{4}{3}G\right)\varepsilon_{22} + \left(K - \frac{2}{3}G\right)\varepsilon_{33} \qquad (2)$$

$$\sigma_{33} = \left(K - \frac{2}{3}G\right)\varepsilon_{11} + \left(K - \frac{2}{3}G\right)\varepsilon_{22} + \left(K + \frac{4}{3}G\right)\varepsilon_{33} \qquad (3)$$

… collecting terms multiplying $\varepsilon_{12}, \varepsilon_{23}, \varepsilon_{13}$

$$\sigma_{12} = 2G\varepsilon_{12}$$

$$\sigma_{23} = 2G\varepsilon_{23}$$

$$\sigma_{13} = 2G\varepsilon_{13}$$

Complete stress-strain relation

$$\sigma_{11} = \left(K + \frac{4}{3}G \right)\varepsilon_{11} + \left(K - \frac{2}{3}G \right)\varepsilon_{22} + \left(K - \frac{2}{3}G \right)\varepsilon_{33} \qquad (1)$$

↓

$$c_{1111} = K + \frac{4}{3}G$$

$$c_{1122} = K - \frac{2}{3}G = c_{1133}$$

Complete stress-strain relation

$$\sigma_{22} = \left(K - \frac{2}{3}G\right)\varepsilon_{11} + \left(K + \frac{4}{3}G\right)\varepsilon_{22} + \left(K - \frac{2}{3}G\right)\varepsilon_{33} \qquad (2)$$

$$\downarrow$$

$$c_{2211} = K - \frac{2}{3}G = c_{2233}$$

$$c_{2222} = K + \frac{4}{3}G$$

Complete stress-strain relation

$$\sigma_{33} = \left(K - \frac{2}{3}G\right)\varepsilon_{11} + \left(K - \frac{2}{3}G\right)\varepsilon_{22} + \left(K + \frac{4}{3}G\right)\varepsilon_{33} \qquad (3)$$

$$\downarrow$$

$$c_{3311} = K - \frac{2}{3}G = c_{3322}$$

$$c_{3333} = K + \frac{4}{3}G$$

Complete stress-strain relation

$$\sigma_{12} = 2G\varepsilon_{12} \quad \longrightarrow \quad c_{1212} = 2G$$

$$\sigma_{23} = 2G\varepsilon_{23} \quad \longrightarrow \quad c_{2323} = 2G$$

$$\sigma_{13} = 2G\varepsilon_{13} \quad \longrightarrow \quad c_{1313} = 2G$$

All other c_{ijkl} are zero

Summary: Expression of elasticity tensor

$$\begin{cases} c_{1111} = c_{2222} = c_{3333} = K + \dfrac{4}{3}G \\ c_{1122} = c_{1133} = c_{2233} = K - \dfrac{2}{3}G \\ c_{1212} = c_{2323} = c_{1313} = 2G \end{cases}$$

Examples – numerical values

Concrete

K = 14 GPa G = 10 GPa

Quartz (sand, stone..)

K = 27 GPa G = 26 GPa

Steel

K = 200 GPa G = 140 GPa