

1.050 Engineering Mechanics

Lecture 2: Dimensional Analysis and Atomic Explosion

1.050 – Content overview

I. Dimensional analysis

1. On monsters, mice and mushrooms
2. Similarity relations: Important engineering tools

Lectures 1-3
Sept.

II. Stresses and strength

2. Stresses and equilibrium
3. Strength models (how to design structures, foundations.. against mechanical failure)

Lectures 4-15
Sept./Oct.

III. Deformation and strain

4. How strain gages work?
5. How to measure deformation in a 3D structure/material?

Lectures 16-19
Oct.

IV. Elasticity

5. Elasticity model – link stresses and deformation
6. Variational methods in elasticity

Lectures 20-31
Nov.

V. How things fail – and how to avoid it

7. Elastic instabilities
8. Plasticity (permanent deformation)
9. Fracture mechanics

Lectures 32-37
Dec.

1.050 – Content overview

I. Dimensional analysis

Lecture 1: Introduction & Galileo's problem

Lecture 2: Dimensional Analysis and Atomic Explosion

Lecture 3: Dimension analysis and application to engineering structures

II. Stresses and strength

III. Deformation and strain

IV. Elasticity

V. How things fail – and how to avoid it

Discorsi e Dimostrazioni Matematiche intorno a Due Nuove Scienze (1638)

- “We clearly see, by what has been demonstrated, that it is impossible to magnify structures to large dimensions, whether in art or nature; [...] it would be equally impossible to create huge bone structures for humans, horses or other animals that would function normally, unless the material employed was much harder and more resistant than usual [...]. Obviously, if we wish to maintain the same proportions of a normal man in a giant, it would be necessary to find a harder and more resistant material to build his bones, or yet admit that his robustness would be proportionally smaller; as he grew up immeasurably we would see him collapse under his own weight.”

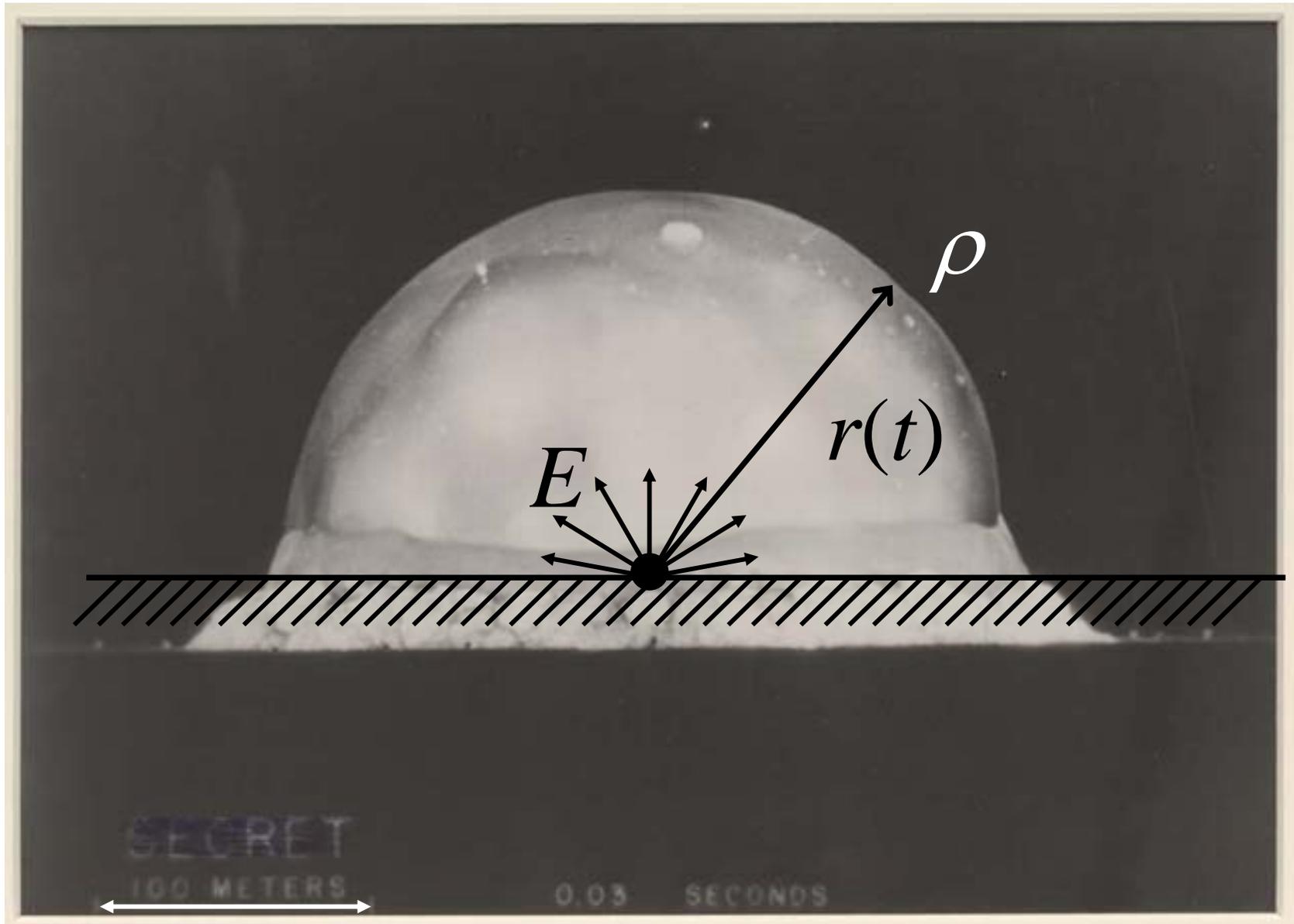
Discorsi e Dimonstrazioni Matematiche intorno a Due Nuove Scienze (1638)

- “another way [...] of making giants or other large animals to live and move like the smaller ones”:
- “this would be possible only by increasing the strength of the bones and also the strength of parts that support the weight and additional loads; but also keeping the same proportions the bone structure would resist only if its specific weight were reduced, as well as the specific weight of the flesh and all other parts supported by the bones.”

$$\underline{\mathcal{N}_{Gal} = \frac{hg\rho}{\sigma_0}}$$

Galileo Number

Exercise: Atomic Explosion



Trinity Test Nuclear Explosion, New Mexico, July 16, 1945 © Library of Congress

Steps of Dimensional Analysis

Recipe

1. Problem Formulation
2. Dimensional Analysis
 1. Build the exponent matrix of dimensions of $N+1$
 2. **Rank of matrix** = k = number of dimensionally independent variables (see next slide)
 3. Choose k independent variables, express $N+1-k$ dimensionless variables
 4. Determine exponents by **solving linear system** (see next+1 slide)
3. Dimensionless expression

Exercise

$$r = f(t, E, \rho) \quad N + 1 = 4$$

	$[r]$	$[t]$	$[E]$	$[\rho]$
L	1	0	2	-3
M	0	0	1	1
T	0	1	-2	0

Here: $k = 3 \Rightarrow N + 1 - k = 1$

$$\Pi_0 = \frac{r}{t^{\alpha_1} E^{\alpha_2} \rho^{\alpha_3}}$$

$$[r] = [t]^{\alpha_1} [E]^{\alpha_2} [\rho]^{\alpha_3}$$

$$\Pi_0 = \frac{r}{t^{2/5} E^{1/5} \rho^{-1/5}} = \text{const}$$

Technique 1: Number of dimensionally independent variables

- Method 1: Look for the maximum number of linearly independent rows or columns
- Method 2: **Rank of a matrix**
 - Manually: Identify the dimension of the biggest sub-square matrix that has a non-zero determinant (math: non-singular)
 - Software: Matlab, Maple, Excel, etc...

Technique 2:

Determination of the exponents

- Find a_1, a_2, a_3 $[r] = [t]^{a_1} [E]^{a_2} [\rho]^{a_3}$
- In a log-representation

$$\begin{array}{l} L \\ M \\ T \end{array} : \begin{pmatrix} \frac{[r]}{1} \\ 0 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} \frac{[t]}{0} \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} \frac{[E]}{2} \\ 1 \\ -2 \end{pmatrix} + a_3 \begin{pmatrix} \frac{[\rho]}{-3} \\ 1 \\ 0 \end{pmatrix}$$

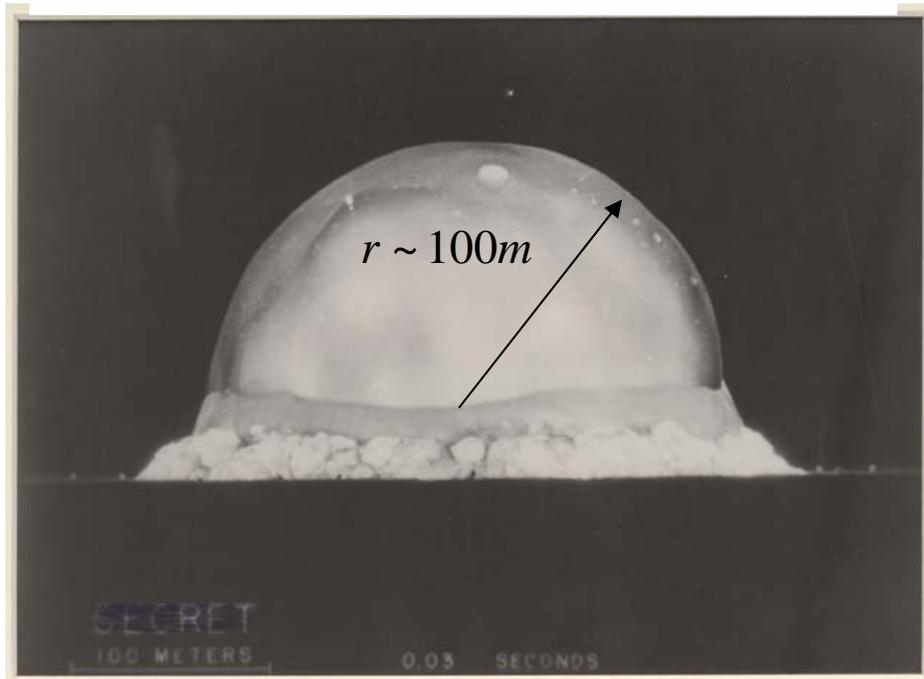
- Linear system: $Ax=y$

$$\begin{array}{l} L \\ M \\ T \end{array} : \begin{pmatrix} \frac{[r]}{1} \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} \frac{[t]}{0} & \frac{[E]}{2} & \frac{[\rho]}{-3} \\ 0 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 1/5 \\ -1/5 \end{pmatrix}$$

G.I. Taylor's Analysis*

- D-Analysis

$$\Pi_0 = \frac{r}{t^{2/5} E^{1/5} \rho^{-1/5}} = \text{const}$$



$t = 30ms$

(*) G.I. Taylor (1950)

- **Top Secret:** What is the energy E released by a nuclear explosion?
- But: High speed photographs were available, giving r and t
- Air density = 2.5 kg/m^3
- $\text{const} \sim O(1)$

$$\underbrace{\frac{5}{2} \log r}_{\text{known}} = \frac{5}{2} \log(\text{const}) + \frac{1}{2} \log \left(\frac{E}{\rho} \right) + \underbrace{\log t}_{\text{known}}$$

G.I. Taylor's Analysis (cont'd)

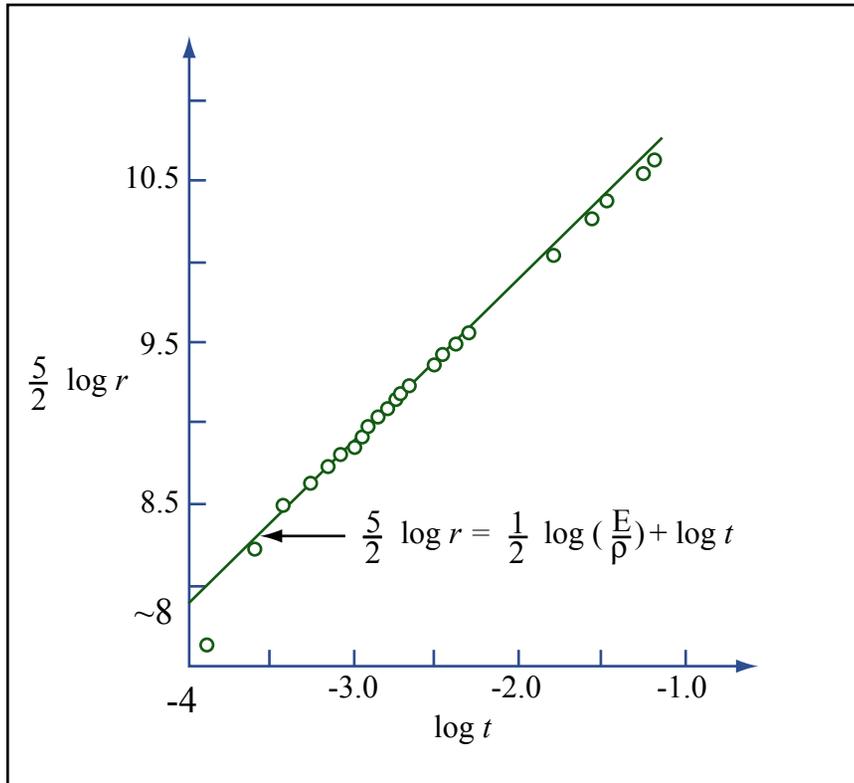


Figure by MIT OpenCourseWare, adapted from Taylor, G. I. "Formation of a Blast Wave by a Very Intense Explosion. II. The Atomic Explosion of 1945." Proceedings of the Royal Society A 201 (1950): 175-186.

- Back Analysis:

$$\frac{5}{2} \log r(t = 10^{-4} s) \sim 8$$

$$\frac{1}{2} \log \left(\frac{E}{\rho} \right) = 8 - \log(10^{-4}) = 12$$

⇓

$$E \sim 10^{21} \text{ erg} = 100,000 \text{ GJ}$$

Comparison: ~ hourly energy production of 20 nuclear power plants

Summary

- **Pi-Theorem**

- Most critical step: problem formulation – if you forget one parameter on which the problem depends, the problem is ill-posed!
- By means of dimensional analysis reduce the complexity of a problem from $N+1$ parameters to $N+1-k$ parameters:
 - Some technique: Exponent matrix – linear system
 - Of critical importance for lab testing: instead of $(N)^a$ tests, you only need to carry out $(N-k)^a$ tests
 - Critical for model scaling: Model (e.g. human) and Prototype (e.g. monster) must have the same invariants.
- Best invariants: not unique, some try and error – you can always recombine invariants as power functions of others.
- If $N = k$, **jackpot** – you have the solution (close to a multiplying constant).
- **In the next lecture and recitation:** Applications