1.050 Engineering Mechanics

Lecture 14:
Strength models for beams (II/II)
M-N coupling
Convexity of strength domain

1.050 – Content overview

I. Dimensional analysis

1. On monsters, mice and mushrooms
2. Similarity relations: Important engineering tools

Lectures 1-3
Sept.

II. Stresses and strength

2. Stresses and equilibrium
3. Strength models (how to design structures, foundations.. against mechanical failure)

Lectures 4-15
Sept./Oct.

III. Deformation and strain

4. How strain gages work?
5. How to measure deformation in a 3D structure/material?

Lectures 16-19
Oct.

IV. Elasticity

5. Elasticity model – link stresses and deformation
6. Variational methods in elasticity

Lectures 20-31
Nov.

V. How things fail – and how to avoid it

7. Elastic instabilities
8. Plasticity (permanent deformation)
9. Fracture mechanics

Lectures 32-37
Dec.

1.050 – Content overview

I. Dimensional analysis

II. Stresses and strength

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Lecture 8: Beam stress model
Lecture 9: Beam model II and summary
Lecture 10: Strength models: Introduction (1D)
Lecture 11: Mohr circle – strength criteria 3D
Lecture 12: Application – soil mechanics: How to build sandcastles
Lecture 13: Strength criterion in beams (I/II)
Lecture 14: Strength criterion in beams (II/II) – convexity of strength domain
Lecture 15: Closure strength models & review for quiz

III. Deformation and strain

IV. Elasticity

V. How things fail – and how to avoid it

Quiz I

- **Wednesday, October 17 in class**
- Please be on time
- Covers first 15 lectures
- Open book

- **Preparation:**
  - Lecture material, PSs, recitation
  - Old quizzes (posted) instead of PS this week
  - Alberto will work through one example (nanoindentation) in recitation
  - Study old quizzes before recitation this week

Strength models

- Equilibrium conditions “only” specify statically admissible stress field, without worrying about if the stresses can actually be sustained by the material – **S.A.**
  From EQ condition for a REV we can integrate up (upscale) to the structural scale

  Examples: Many integrations in homework and in class; Hoover dam etc.

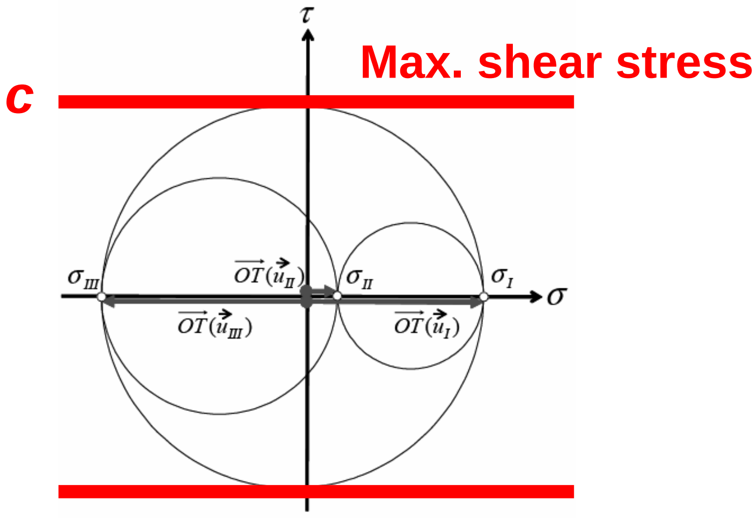
- Strength compatibility adds the condition that in addition to S.A., the stress field must be compatible with the strength capacity of the material – **S.C.**
  In other words, at no point in the domain can the stress vector exceed the strength capacity of the material

  Examples: Sand pile, foundation etc. – Mohr circle

Strength models

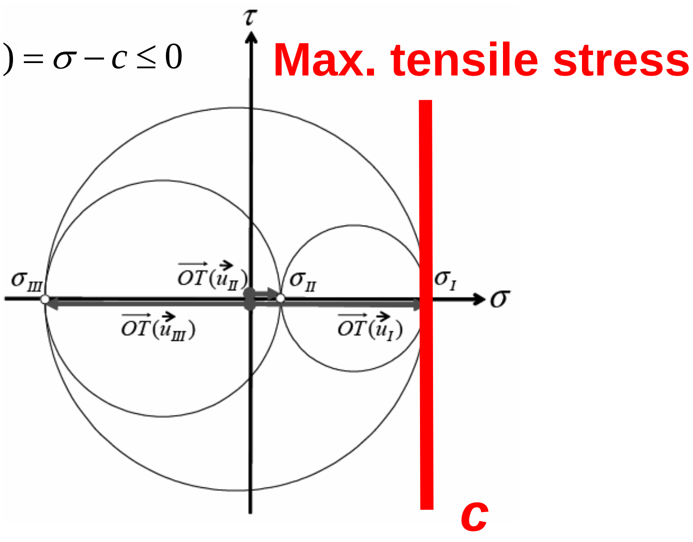
$$\boxed{\text{Tresca: } \forall \overrightarrow{n};\ f\left(\overrightarrow{T}\right) = |\tau| - c \leq 0}$$

$$\text{cohesion, } 2c = \sigma_0$$



Tresca criterion

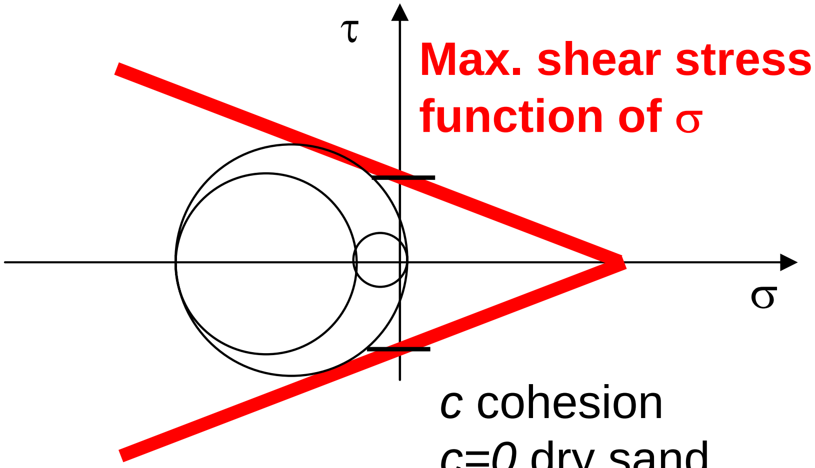
$$\forall \vec{n} : f(\vec{T}) = \sigma - c \leq 0$$



Tension cutoff criterion

Strength models

$$\boxed{\text{Mohr-Coulomb: } \forall \vec{n};\ f\left(\vec{T}\right) = |\tau| + \sigma \underbrace{\tan\varphi}_{=\mu} - c \leq 0}$$



Max. shear stress function of σ

c cohesion

c=0 dry sand

Mohr-Coulomb

Review: Beam models

Beam model: Special case of the general continuum model

Special geometry – highly distorted system (much longer than wide)

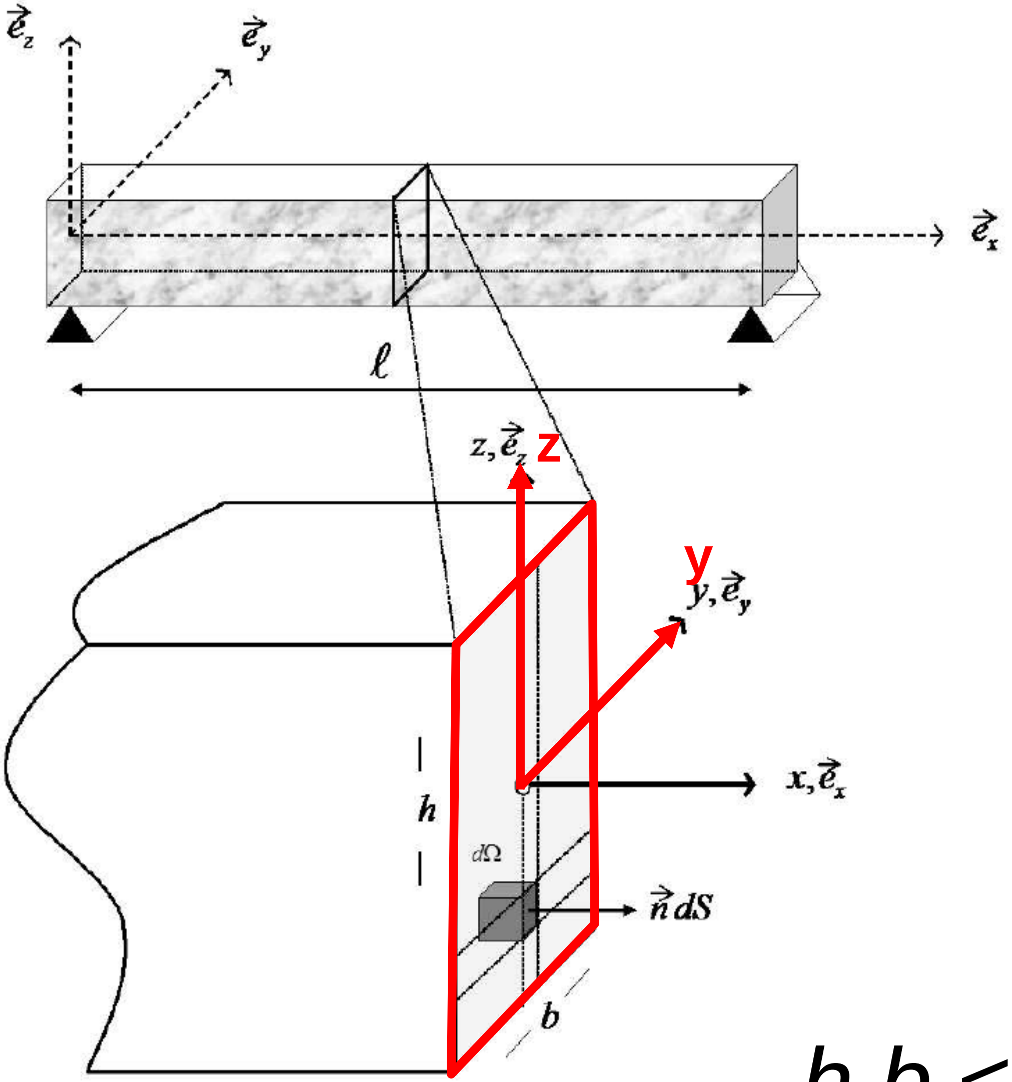
Special form of stress tensor:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & & \text{sym} \\ \sigma_{yx} & \sim 0 & \\ \sigma_{zx} & \sim 0 & \sim 0 \end{pmatrix}$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(x; y, z)$$

For fixed *x* (section choice):

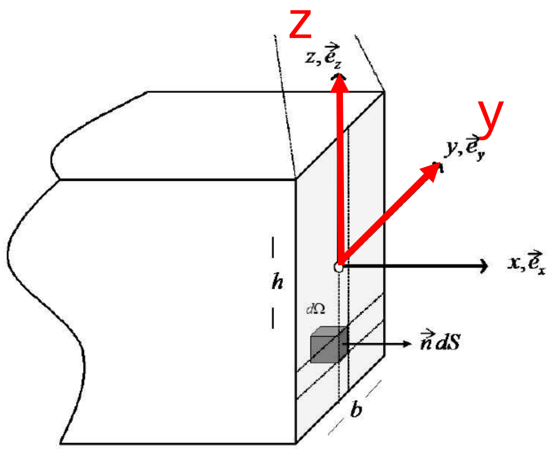
$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(y, z)$$



$h,b << l$

Link between section quantities and section stress field

- Section force and moment distribution is due to a particular stress tensor distribution in the section



y,*z*: C.S. in section

Reduction formulas

dS=dzdy

$$N_x = \int_S \sigma_{xx}(y,z)\; dS$$
$$Q_y = \int_S \sigma_{yx}(y,z)\; dS$$
$$Q_z = \int_S \sigma_{zx}(y,z)\; dS$$

$$\begin{pmatrix} \mathcal{M}_x \\ \mathcal{M}_y \\ \mathcal{M}_z \end{pmatrix} = \int_S \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \times \begin{pmatrix} \sigma_{xx} \\ \sigma_{yx} \\ \sigma_{zx} \end{pmatrix} dS = \begin{pmatrix} \int_S [y\sigma_{zx} - z\sigma_{yx}]\, dS \\ \int_S z\sigma_{xx} dS \\ -\int_S y\sigma_{xx} dS \end{pmatrix}$$

$$\downarrow$$

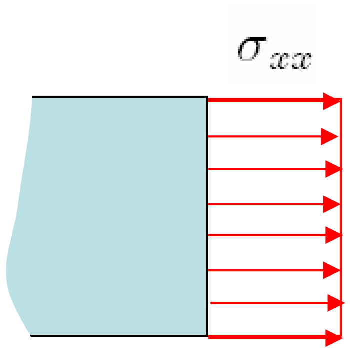
$$\overrightarrow{x}_S = y\,\overrightarrow{e}_y + z\,\overrightarrow{e}_z.$$

Torsion

Bending

Example

Stress distribution in section



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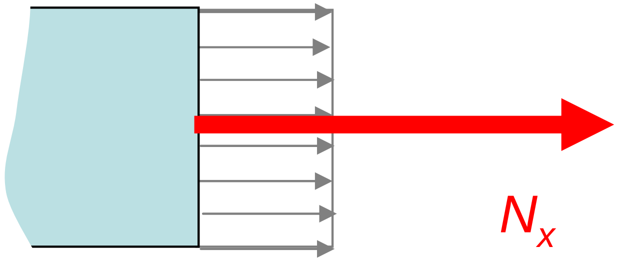
Equivalent normal force

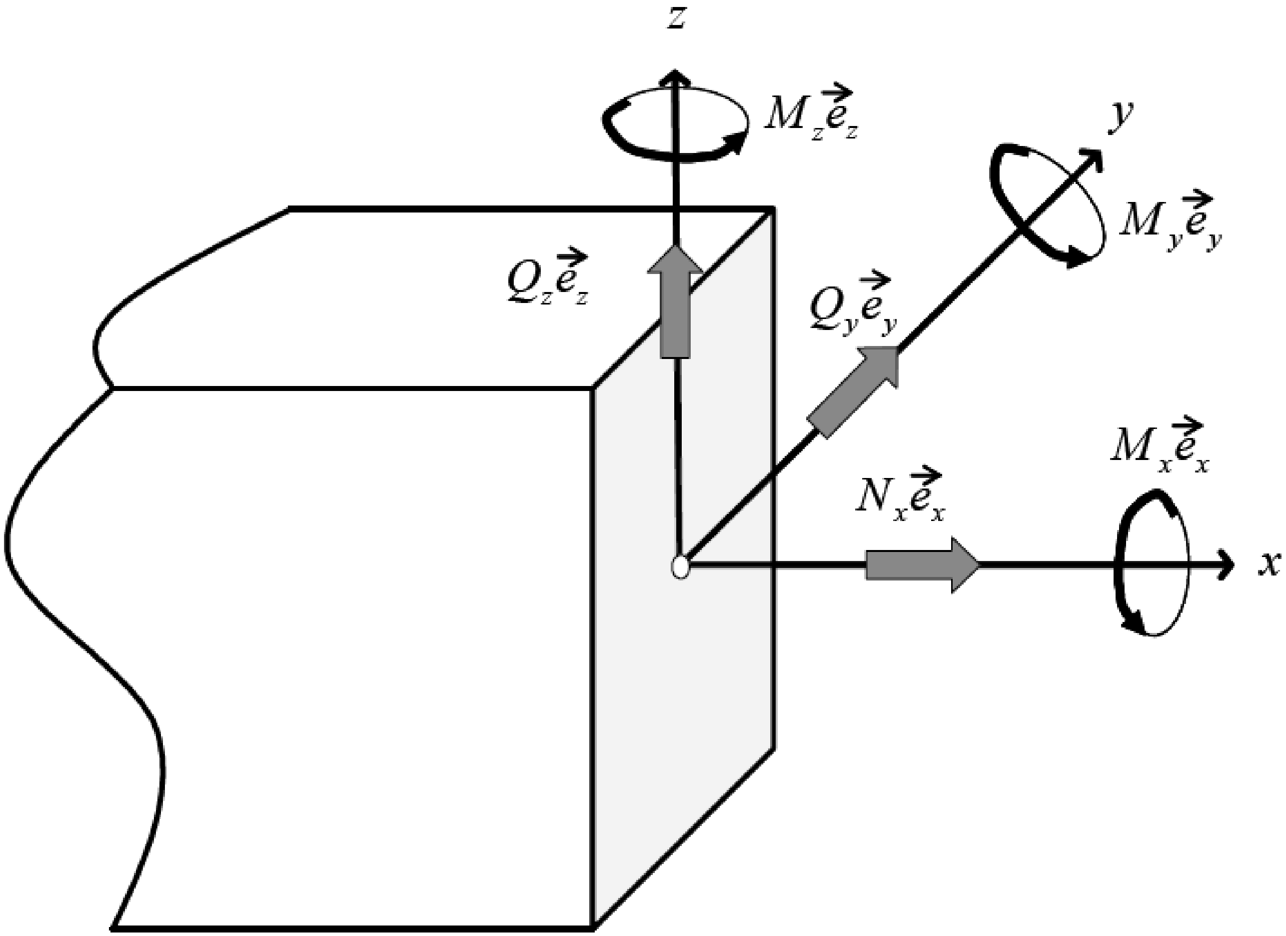


$$N_x = \int_S \sigma_{xx}\left(y, z\right)\, dS$$

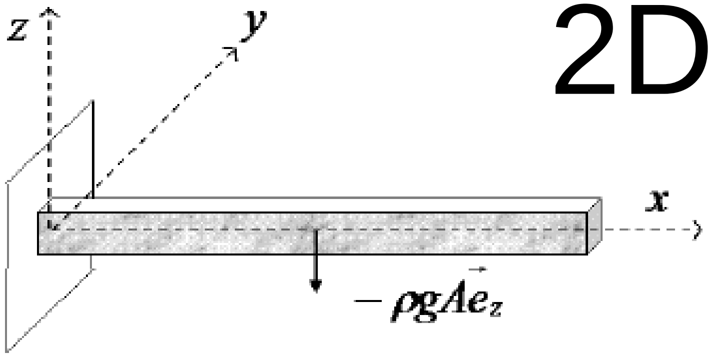
Review: Beam models

	Continuum model	**Beam model**
Differential element	$d\Omega$	dx
Equilibrium condition	$\boxed{\text{in } \Omega : \begin{cases} \vec{T}(\vec{n}) = \boldsymbol{\sigma}\cdot\vec{n} \\ \operatorname{div}\boldsymbol{\sigma} + \rho\,(\vec{g} - \cancelto{0}{\vec{a}}) = 0 \\ \sigma_{ij} = \sigma_{ji} \end{cases}}$ $\boxed{\text{on } \partial\Omega;\ \vec{T}^{\,d} = \vec{T}(\vec{n})}$	$\forall x \in [0,\ell]\,;\ \frac{d\vec{F}_S}{dx} + \vec{f}^{\,ext} = 0$ $\forall x \in [0,\ell]\,;\ \frac{d\vec{\mathcal{M}}_S}{dx} + \vec{e}_x \times \vec{F}_S = 0$ $\boxed{x = 0,\ell : \begin{cases} \vec{F}^{d} = \vec{F}_S(x) \\ \vec{\mathcal{M}}^{d} = \vec{\mathcal{M}}_S(x) \end{cases}}$ $\vec{f}^{\,ext} = f_x\vec{e}_x + f_y\vec{e}_y + f_z\vec{e}_z$ Line force density $\left[\vec{f}^{\,ext}\right] = [F]\,L^{-1}$
Simplification	Hydrostatics (fluid): $\boxed{\text{in } \Omega : \begin{cases} \vec{T}(\vec{n}) = -p\,\vec{n} \\ \boldsymbol{\sigma} = -p\mathbf{1} \\ -\operatorname{grad} p + \rho\,\vec{g} = 0 \end{cases}}$ +BCs $-\frac{\partial p}{\partial x} = 0;\ -\frac{\partial p}{\partial y} = 0;\ -\frac{\partial p}{\partial z} - \rho g = 0$	2D: $\boxed{\begin{array}{ll} \frac{dN_x}{dx} + f_x = 0 & \frac{dQ_z}{dx} + f_z = 0 \\ \frac{d\mathcal{M}_y}{dx} - Q_z = 0 & \end{array}}$ +BCs 

Beam models: Moments



2D Example

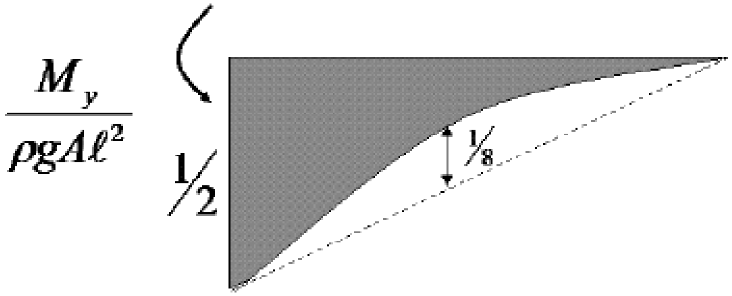


Cantilever beam
Dead weight (gravity)

EQ and solution

$$\forall x \in [0,\ell]\,;\ \begin{cases} \dfrac{dN_x}{dx} = 0 \quad \Rightarrow N_x = C_1 \\ \dfrac{d\mathcal{M}_y}{dx} - \overbrace{(C_3 + \rho g S x)}^{Q_z} = 0 \quad \Rightarrow \mathcal{M}_y = \underline{C_5} + \underline{C_3} x + \frac{1}{2}\rho g A x^2 \\ \dfrac{dQ_z}{dx} - \rho g A = 0 \quad \Rightarrow Q_z = \underline{C_3} + \rho g A x \end{cases}$$

$$\boxed{\begin{array}{l} \overrightarrow{F}^d(\ell) = 0;\ \overrightarrow{\mathcal{M}}^d(\ell) = 0 \\ Q_z(x) = \rho g A (x - \ell)\,;\ \mathcal{M}_y(x) = \frac{1}{2}\rho g A (\ell - x)^2 \end{array}}$$



Example: Coupled M-N strength domain