

1.050: Continuum Strength Models (HW#5)

Due: October 10, 2007

MIT – 1.050 (Engineering Mechanics I)

Fall 2007

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Team Building and Team Work: We strongly encourage you to form Homework teams of three students. Each team only submits one solution for correction. We expect true team work, i.e. one where everybody contributes equally to the result. This is testified by the team members signing at the end of the team copy a written declaration that "the undersigned have equally contributed to the homework". Ideally, each student will work first individually through the homework set. The team then meets and discusses questions, difficulties and solutions, and eventually, meets with TA or instructor. Important: Specify all resources you use for your solution.

The following set of exercises is designed to train you in the use of equilibrium and strength models for continuum systems. For each exercise, show us how you came to your answer and result. We highly encourage you to make drawings where appropriate.

1. **Stress state and strength criterion:** We consider the next two stress tensors (parameter $p > 0$), given in a Cartesian system of coordinates of basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$:

$$\underline{\underline{\sigma}} = p \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad ; \quad \underline{\underline{\sigma}} = p \begin{bmatrix} 0 & -2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

For each stress tensor, obtain:

- The eigenvalues and eigenvectors.
- Display the Mohr circles.
- Find the minimal value of p at which the material fails for a material governed by a Tresca strength criterion of cohesion $c = 1$.
- Find the two failure planes (in the $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ basis) and the stress vectors that act on them.
- What is the underlying assumption for the Tresca strength model?
- Answer the same questions (a through d) in the case of a material governed by a Mohr-Coulomb criterion of cohesion $c = 1$ and of friction angle $\varphi = \pi/5$.

2. A **nano truss**: Advances in nanoscience and nanotechnology* now enable us to build structures at molecular scales, with atomistic precision. An important building block of many nanostructures are carbon nanotubes (CNTs), a particularly sturdy form of carbon arranged in a tubular structure. A picture of a CNT is shown below.

From a mechanical perspective, CNTs are particularly intriguing as they are one of the strongest and stiffest materials known; yet, they are extremely lightweight. Applications of CNTs range from reinforcement for concrete (to supplement or replace steel reinforcement), polymers to important applications in energy related technologies, such as hydrogen storage for fuel cell applications. Nanoscopic truss structures of CNTs may find applications in ultra-small devices to clean water, to harvest sunlight or to analyze genetic information.

This exercise introduces you to the possibility to utilize structural engineering tools typically applied to ‘macroscopic’ objects like bridges, buildings etc. to nanostructured objects.

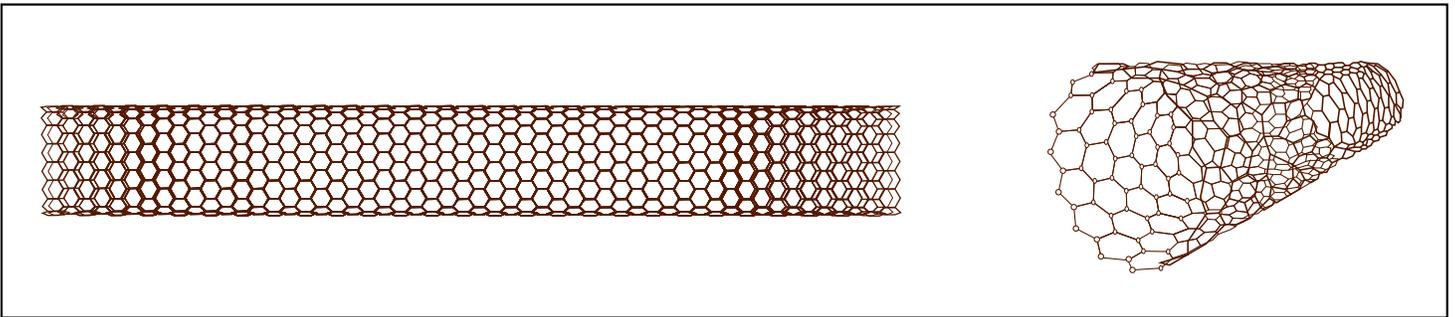


Figure by MIT OpenCourseWare.

Snapshots of a CNT. It can be envisioned as a rolled up sheet of graphene (graphene denotes a single layer in graphite). Many different types of CNTs (different sizes, multiple nested CNTs like in a Russian Doll...) can be synthesized.

Here we consider a truss structure build out of CNT elements. We will consider a particular geometry of a CNT as shown on the next page. Below you will find some relevant information about the material and the geometry of CNTs:

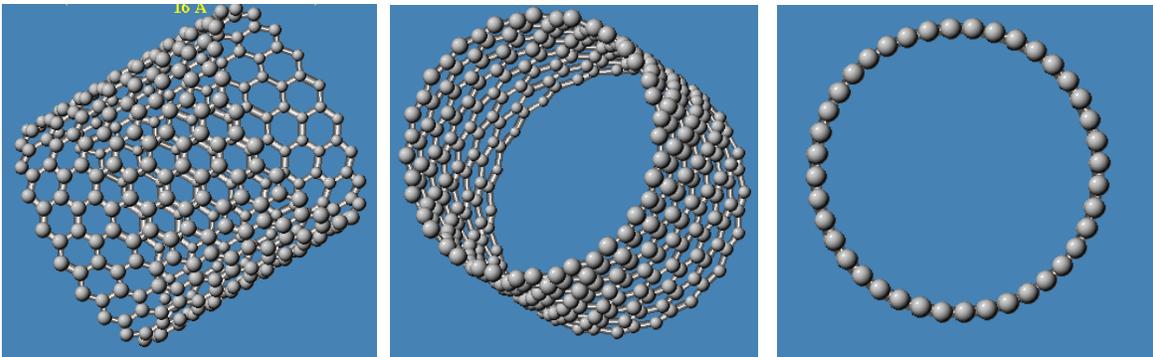
- Young’s modulus (a measure of stiffness), $E = 2 \text{ TPa}$
- Bond strength of a single covalent C-C bond, $P_{bond} = 3 \text{ nN}$
- Outside radius of CNT, $R_0 = 8 \text{ \AA}$
- Area moment of inertia (assuming circular cross-section), $I = \frac{\pi R^4}{4}$
- Critical buckling load for a simply supported column (which we had applied earlier to our truss members), $P_{crit} = \frac{EI\pi^2}{L^2}$, where I is the area moment of inertia, and L is the length of the member.

* **Richard Feynman** (Professor at Caltech and Nobel Laureate) is often considered the founding father of nanoscience and nanotechnology. In a famous speech in 1959 he first articulated the possibility of controlling matter at atomistic and molecular scale and utilizing this for technological applications (<http://www.rpi.edu/dept/materials/COURSES/NANO/shaw/Page2.html>). Over the past decades, many advances have been made in this field, among others, a new form of carbon has been discovered, arranged in a tube-like structure, referred to as ‘carbon nanotubes’. Some of Feynman’s visions have become a reality, while many aspects still remain a research topic. **Eric Drexler**, an alumnus of MIT, is a pioneer in the implementation of such ideas and was maybe the first ‘nanoengineer’.

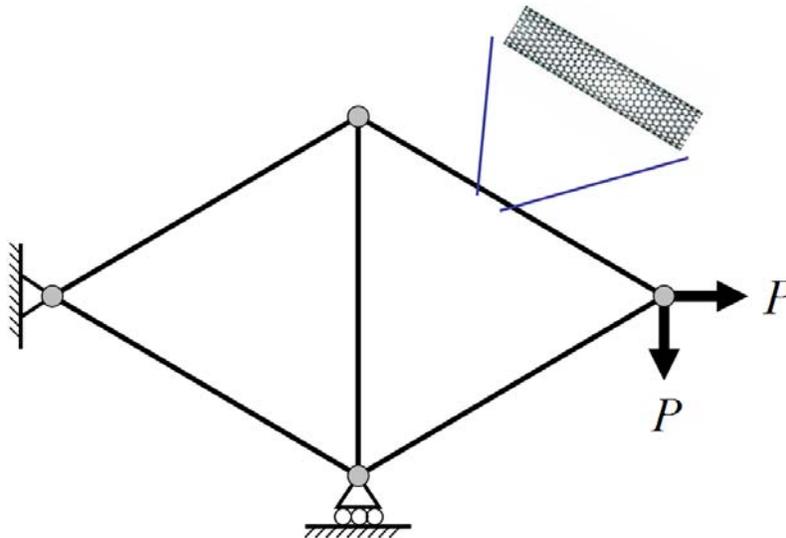
- Both triangles defined in the nano-truss are isosceles, with side lengths $L = 10$ nm
- Loads applied on the nano-truss, $P = 20$ nN

For this problem, perform the following tasks:

- Compute the axial forces in each member of the nano-truss.
- For the given geometry, material, properties, and loading condition, determine the robustness for each member (hint: consider a range of possible failure modes for each member). Derive all material parameters from the information given above.
- Given the analysis in part (b), what could be a critical condition to be optimized to ensure optimal utilization of the material? Outline several steps towards that goal, e.g. changes of geometry, the structure of the CNT and others.



A selection of views of the carbon nano tube (CNT) that is used to build the truss structure. Images generated in Center of Computational Science web application, University of Kentucky, <https://www.ccs.uky.edu/~ernst/carbontubes/structure.html> (you could use this tool to create your own CNT structures!)

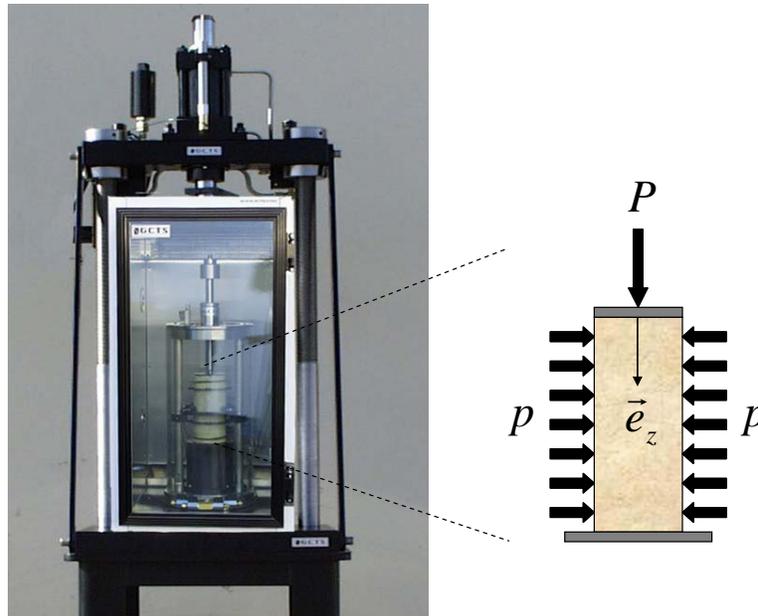


Sketch of the nano-truss. All members have the same length. The CNTs are connected by molecular structures that represent frictionless joints.

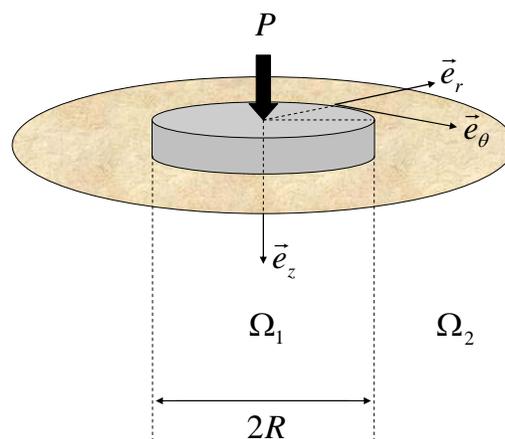
3. **Limit Load Below a Circular Foundation:** Sketch (b) shows a soil continuum below a circular foundation of radius R , subjected to the foundation load P . We want to evaluate the maximum load the soil substrate can support.

Throughout this exercise (parts (a) and (b)), we make the following simplifying assumptions:

- All contacts are *frictionless*.
- The weight of the soil is neglected.
- We adopt cylindrical coordinates of basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$, see sketch.



GCTS Testing Systems (2007). Universal testing machine, <http://www.gcts.com/images/photos/173.jpg>
 Courtesy of GCTS Testing Systems. Used with permission.



Figures for Problem 3. a Triaxial test experiment (top)
b Soil continuum below a circular foundation (bottom).

- a. **Triaxial Test:** In order to characterize the soil behavior, it is common engineering practice to core a cylindrical sample of radius R and height H . The sample is brought into a lab, where they are subjected to the so-called *triaxial test*. The test consists of placing the cylindrical specimen between two rigid plates of a material testing machine (see figure). The lower surface $z = H$ is fixed. The upper plateau ($z = 0$) exerts a vertical force $\vec{F}_{ext} = P\vec{e}_z$ (with $P > 0$) on the specimen surface. In addition to the vertical force, a pressure p is exerted on the cylinder wall. Both P and p are increased until the failure is reached. Throughout this exercise (parts (a) and (b)) we assume $P > \pi p R^2$.
- Statically Admissible Stress Field:* For the triaxial test, specify precisely the conditions that statically admissible stress fields need to satisfy. Among all possible statically admissible stress fields, determine one that is constant in the sample (Reminder: use cylinder coordinates).
 - Mohr-Stress-Plane:* In the (σ, τ) Mohr stress plane, display the chosen constant stress field. In both the Mohr plane and the material plane, determine the surface and the corresponding stress vector, where the shear stress is maximum.
 - Mohr-Coulomb material:* The triaxial tests reveal that the material obeys a Mohr-Coulomb strength criterion, defined by the cohesion c and the friction angle φ . Display the strength criterion in the Mohr-stress plane. Determine the relationship that links P and p when the material reaches the strength limit. In both Mohr plane and material space, determine the orientation of the surface(s) of the sample where the material reaches the strength limit.
- b. **From the Triaxial Test to the Limit Load of a Circular Foundation:** With the strength values in hands, we turn to the circular foundation problem. For purpose of analysis, we divide the substrate in two domains Ω_1 and Ω_2 , whose interface is situated at $r = R$.
- Statically Admissible Stress Field:* In both domains we assume constant stress fields, of the form:

$$\underline{\underline{\sigma}}^{(i)} = a^{(i)}\vec{e}_r \otimes \vec{e}_r + b^{(i)}\vec{e}_\theta \otimes \vec{e}_\theta + c^{(i)}\vec{e}_z \otimes \vec{e}_z$$

- where $a^{(1)}, b^{(1)}, c^{(1)}$ and $a^{(2)}, b^{(2)}, c^{(2)}$ are six constants which you need to determine so that the stress field is statically admissible everywhere in $\Omega = \Omega_1 + \Omega_2$. *Hint:* For domain Ω_1 (which is situated below the foundation), you may want to consider the triaxial stress field from part (a). Then check how to extend this stress field into Ω_2 .
- Mohr-Stress-Plane:* Display the stress fields of Ω_1 and Ω_2 in the Mohr-stress plane.
 - Mohr-Coulomb material:* In the Mohr-Stress plane display the Mohr-Coulomb criterion. For the two domains, determine the stress vector components (σ, τ) where the material reaches the strength criterion. On this basis, determine the maximum load P the soil substrate can support without violating neither equilibrium nor Mohr-Coulomb strength criterion. Determine the orientation of the surfaces, e.g. the angle relative to vertical, where the material reaches the strength criterion.