Probabilistic Planning 2

Nathaniel Osgood

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Topics

- PERT (Cont'd)
  - Review
  - Merge node bias
  - PNet refinement
- Monte Carlo
- Simulation approaches
  - General
  - Demo
  - Process Interaction
  - Activity Scanning

PERT Basics

- Expresses uncertainty in *activity* duration
  - Beta distribution assumed for activities
- Assume normally distributed *project* duration
  - Project Duration Tends to be Normally Distributed (approx. sum of random variables)
  - Assumes Independent Activity Durations - Not Always Satisfied

Stochastic Approach

- Optimistic a
- *Most Likely (mode – not mean)* m
- Pessimistic b
- Expected Duration $\bar{d} = \frac{1}{3}\left[2m + \frac{1}{2}(a+b)\right] = \frac{a+4m+b}{6}$
- Variance $v = s^2$
- Standard Deviation $s = \frac{b-a}{6}$

Recall: Steps in PERT Analysis

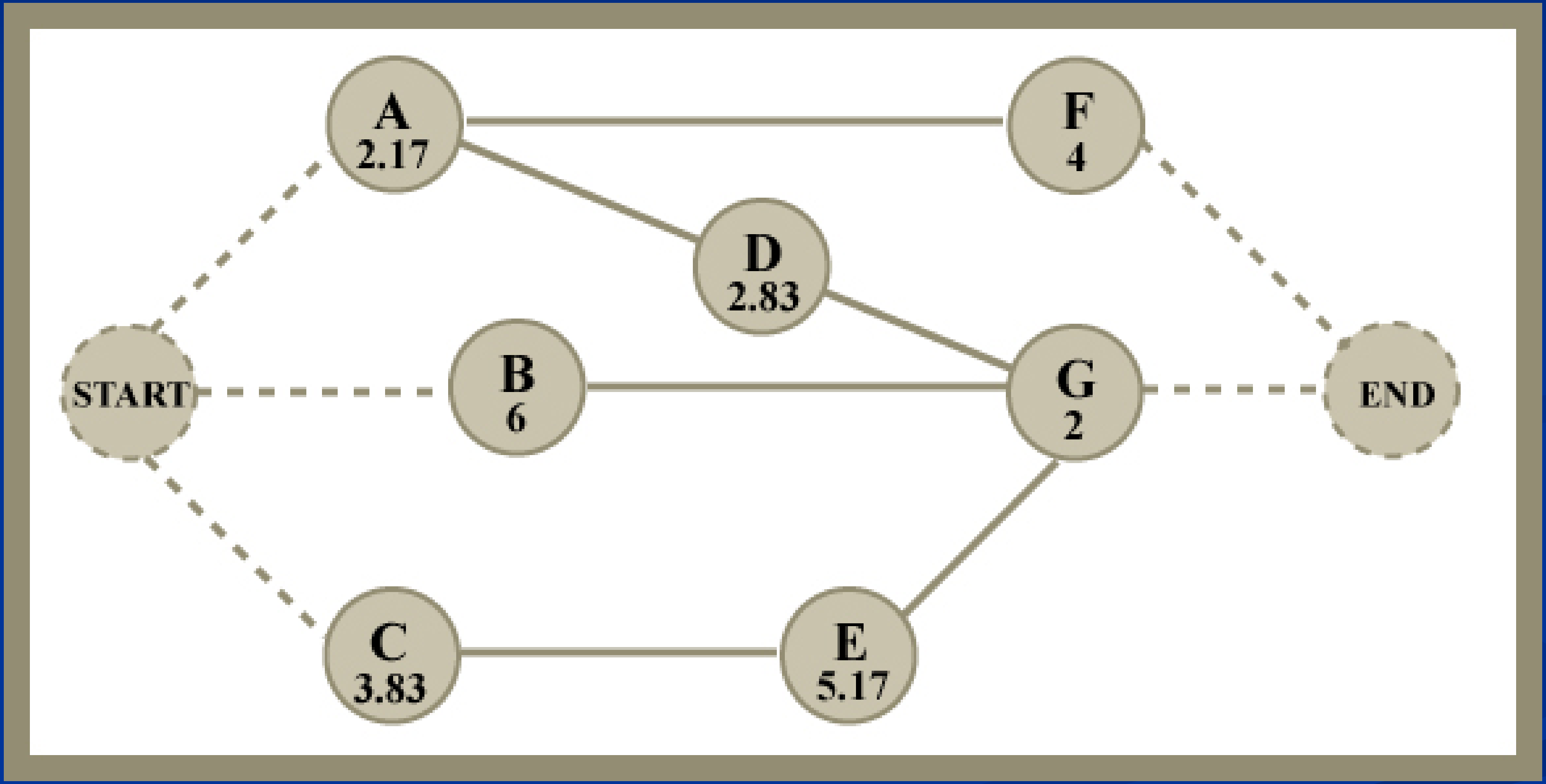
- For each activity k
  - Obtain a_k, m_k (mode) and b_k
  - Compute expected activity duration (mean) $d_k=t_e$
  - Compute activity variance $v_k=s^2$
- Compute expected project duration $D=T_e$ using standard CPM algorithm
- Compute Project Variance $V=S^2$ as sum of critical path activity variance (*this assumes independence!*)
  - In case of multiple critical paths use the one with the largest variance
- Compute probability complete project by time t
  - Assuming project duration normally distributed

PERT Example

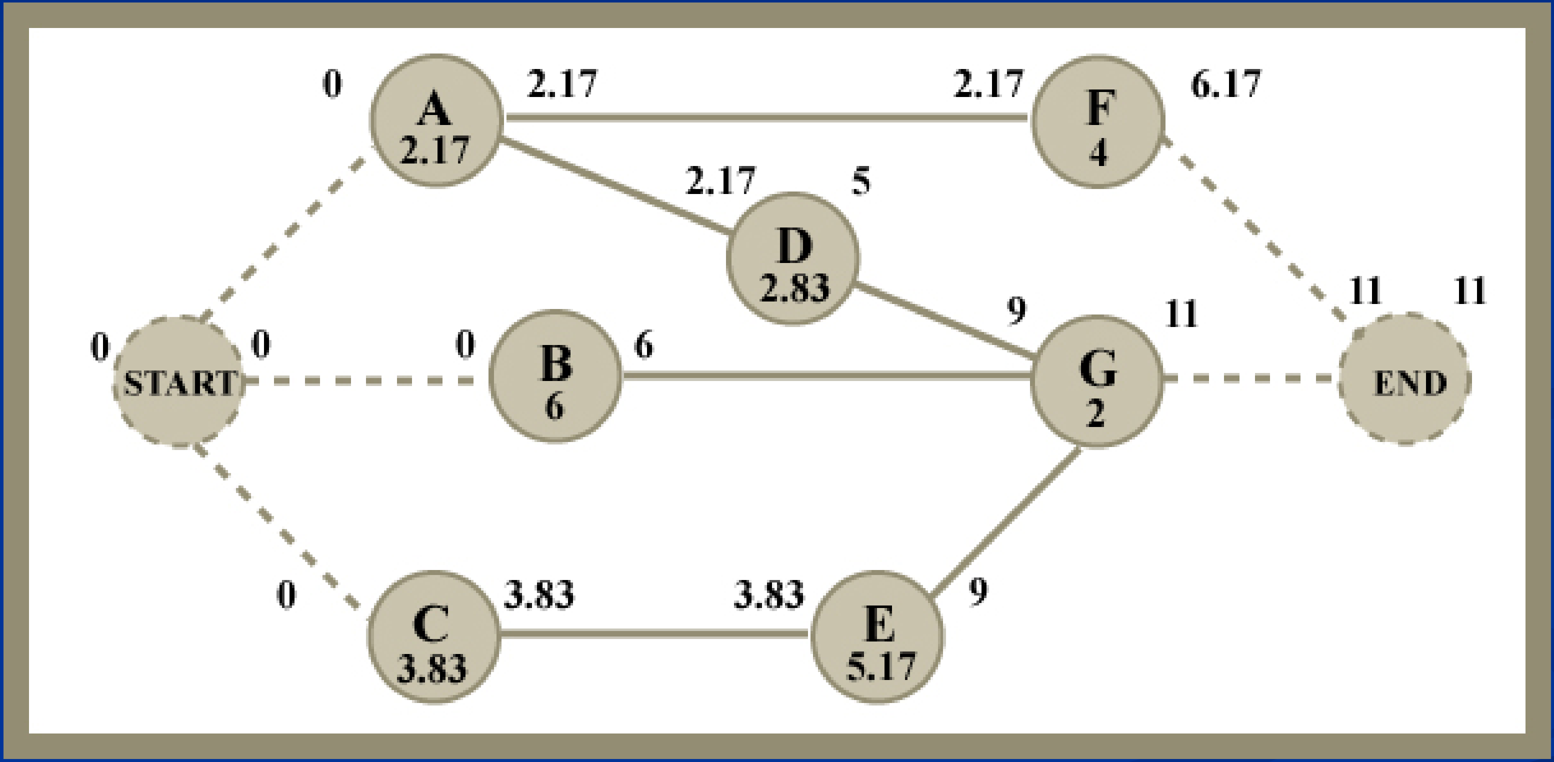
Calculated

Activity	Predecessor	a	m	b	d	v
A	-	1	2	4	2.17	0.25
B	-	5	6	7	6.00	0.11
C	-	2	4	5	3.83	0.25
D	A	1	3	4	2.83	0.25
E	C	4	5	7	5.17	0.25
F	A	3	4	5	4.00	0.11
G	B,D,E	1	2	3	2.00	0.11

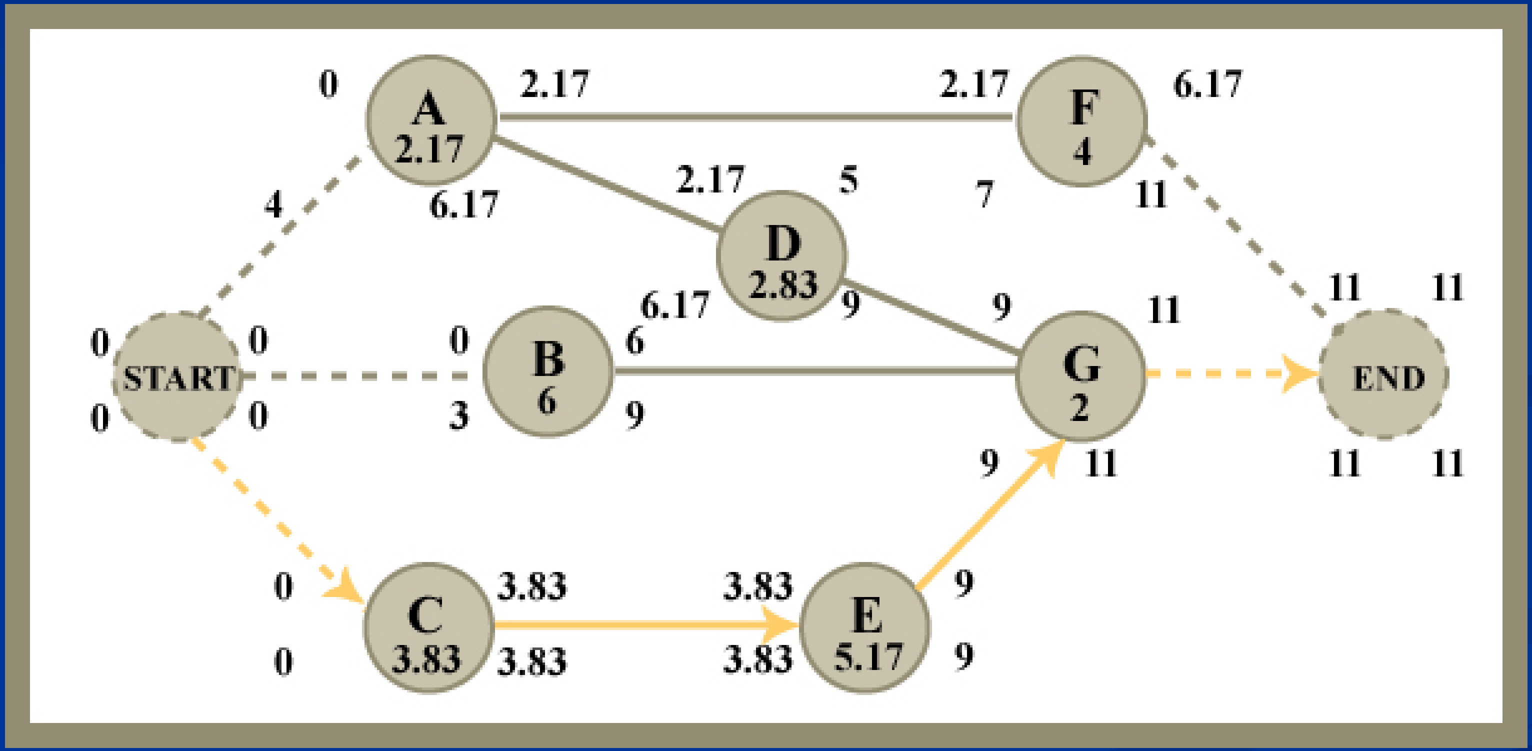
Activity on Node Example



Forward Pass



Backward Pass



PERT Example-Standard Deviation

$$T_e = 11$$

$$S^2 = V[C] + V[E] + V[G]$$

$$= 0.25 + 0.25 + 0.1111$$

$$= 0.6111$$

$$S = \sqrt{0.6111}$$

$$= 0.7817$$

PERT Analysis-Probability of Ending before 10 (Critical Path Only)

$$
\begin{aligned}
P\left(T \le T_d\right) = P(T \le 10) \\
&= P\left(z \le \frac{10 - T_e}{S}\right) \\
&= P\left(z \le \frac{10 - 11}{0.7817}\right) \\
&= P\left(z \le -1.2793\right) \\
&= 1 - P\left(z \le 1.2793\right) \\
&= 1 - 0.8997 \\
&= 0.1003 \\
&= 10\%
\end{aligned}
$$

PERT Analysis - Probability of Ending before 13 (Critical Path Only)

$$P(T \le 13) = P\left(z \le \frac{13-11}{0.7817}\right)$$

$$= P(z \le 2.5585)$$

$$= 0.9948$$

PERT Analysis - Probability of Ending between 9 and 11.5(CP Only)

$$
\begin{aligned}
P\left(T_L \le T \le T_U\right) &= P\left(9 < T \le 11.5\right) \\
&= P\left(T \le 11.5\right) - P\left(T \le 9\right) \\
&= P\left(z \le \frac{11.5 - 11}{0.7817}\right) - P\left(z \le \frac{9 - 11}{0.7817}\right) \\
&= P\left(z \le 0.6396\right) - P\left(z \le -2.5585\right) \\
&= P\left(z \le 0.6396\right) - \left[1 - P\left(z \le 2.5585\right)\right] \\
&= 0.7389 - \left[1 - 0.9948\right] \\
&= 0.7389 - 0.0052 \\
&= 0.7337
\end{aligned}
$$

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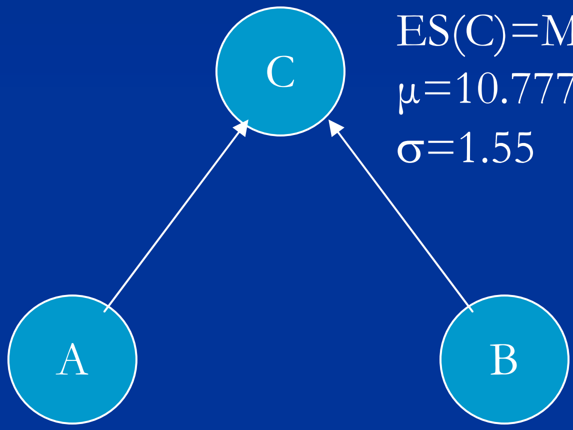
Merge Node Bias

- Misleading to consider only *variance* from single predecessor for each node on critical path
  - Early start of node depends on *maximum* of finish (or start) times of predecessors – including non-critical!
- Basically ES = RV that is max of (non-iid) RVs
- Effect stronger if have
  - More predecessors
  - Predecessors with almost equal timing
  - Less dependency among predecessors
- Consequence: *Unrealistic optimism* with respect to expected completion times, but especially *variance*

Example Merge Node



ES(C)=Max(EF(A), EF(B))
$\mu=10.777$
$\sigma=1.55$

Late Finish: N(10,1)   Late Finish: N(9,3)

Sample Problem

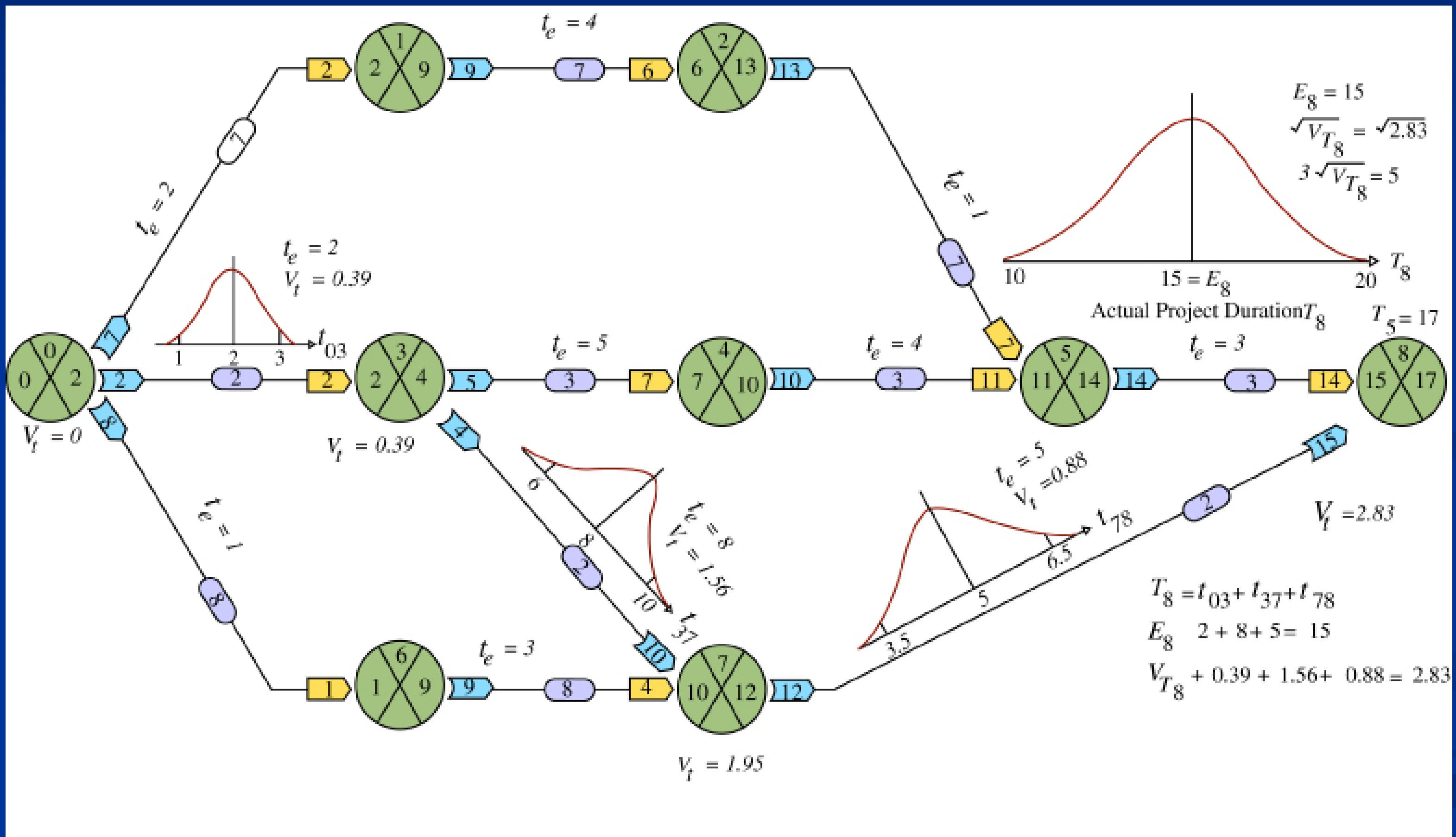


Illustration of the "conventional" PERT statistical approach to the network

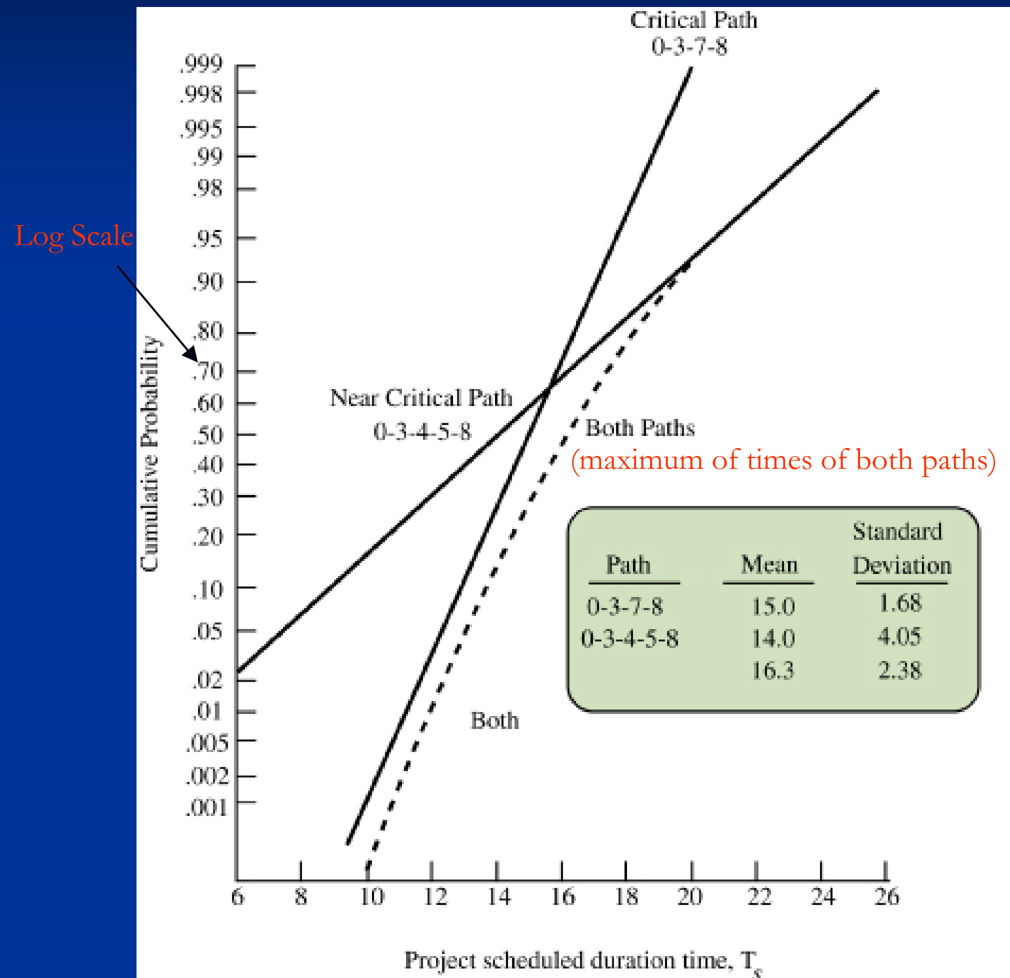
Derived Parameters

MEAN AND STANDARD DEVIATION OF THE CRITICAL AND NEAR CRITICAL PATHS FOR NETWORK

ACTIVITY	TIME ESTIMATES			PATH 0-3-7-8 (Critical Path)		PATH 0-3-4-5-8	
	a	m	b	MEAN	VARIANCE	MEAN	VARIANCE
0-3	1	2	3	2	0.39	2	0.39
3-7	6	8	10	8	1.56	-	-
7-8	3.5	5	6.5	5	0.88	-	-
3-4	1	4	13	-	-	5	14.06
4-5	2	4	6	-	-	4	1.56
5-8	2	3	4	-	-	3	0.39
TOTALS*				15.0	2.83	14.0	16.40
STANDARD DEVIATION				-	1.68	-	4.05

* The mean and variance of the duration of a path is merely the sum of the means and variances of the activities along the path in question; the standard deviation of the path duration is then obtained as the square root of its variance.

Impact of Multiple Paths



Path	Mean	Standard Deviation
0-3-7-8	15.0	1.68
0-3-4-5-8	14.0	4.05
	16.3	2.38

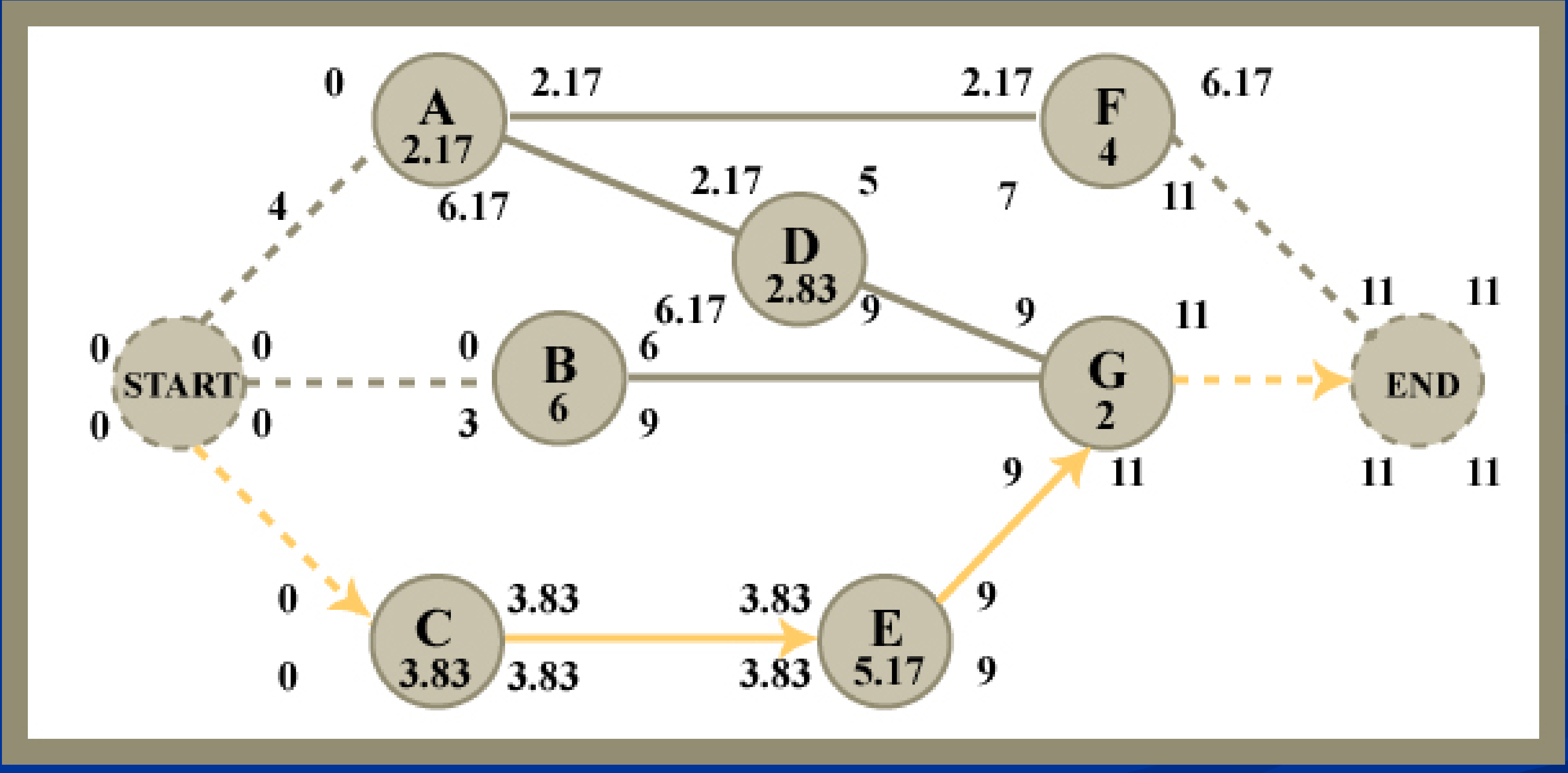
Cumulative probability of path and project completion on or before time T_S

Naïve Approach

- Consider variance from *all* paths entering a merge node

- Assume Probability EF(i)<T = $\prod_{j \in \text{Paths To(i)}} P(EF(j)<T)$

Recall


START
A
2.17
B
6
C
3.83
D
2.83
E
5.17
F
4
G
2
END
0
4
6.17
2.17
5
7
11
9
3
3.83
6.17
11

PERT Analysis - ADG Path

$$T_e = 7$$

$$S^2 = V[A] + V[D] + V[G]$$

$$= 0.25 + 0.25 + 0.11$$

$$= 0.6111$$

$$S = \sqrt{0.6111}$$

$$= 0.7817$$

PERT Analysis - ADG Path Probability of Ending before 10

$$P(T \le 10) = P\left(z \le \frac{10-7}{0.7817}\right)$$

$$= P(z \le 3.8378)$$

$$= 0.9999$$

PERT Analysis - BG Path

$$T_e = 8$$

$$S^2 = V[B] + V[G]$$

$$= 0.1111 + 0.1111$$

$$= 0.2222$$

$$S = \sqrt{0.2222}$$

$$= 0.4714$$

PERT Analysis - BG Path Probability of Ending before 10

$$P(T \le 10) = P\left(z \le \frac{10-8}{0.4714}\right)$$

$$= P(z \le 4.2429)$$

$$= 0.9999$$

PERT Analysis - ADG , BG and CEG Paths Probability of Ending before 10

$$P_c(T \le 10) = P(T_{CEG} \le 10)P(T_{ADG} \le 10)P(T_{BG} \le 10)$$
$$= (0.1003)(0.9999)(0.9999)$$
$$= 0.1003$$
$$= 10\%$$

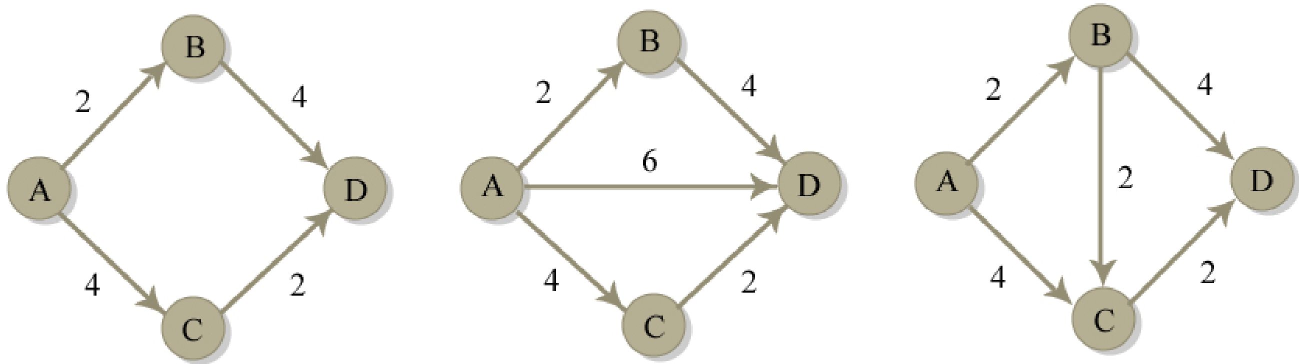
PERT (cont):

- For the G finish within 10 days, all 3 paths must finish in 10 days or less (i.e. ADG and CEG and BG)
- Calculated as:

  $P(T \leq 10) = P(ADG \leq 10) * P(CEG \leq 10) * P(BG \leq 10)$
- What is wrong with this equation?
- The equation assumes the path durations are independent!
- This cannot be if there are shared activities between the paths.

Example of Multiple Paths – Dependent and Independent

Effect of Parallel Paths, with and without Correlation, on the Merge Event Bias



PERT (mean)	6.00	6.00	6.00
Exact (mean)	6.89	7.34	7.07
Error (mean)	12.9%	18.2%	15.2%

Activities with duration 2 have σ=.707

Activities with duration 4 have σ=1.414

PERT (cont):

- A Solution: Use either
  - PNet
  - Monte Carlo simulation

PNet

- Aims at addressing merge node bias
- Basically works by
  - Enumerate all paths P s.t. Dur(P)> αDur(crit path)
  - Rank paths by decreasing duration (by decreasing naively-estimated variance for ties)
  - Compute linear correlation coefficient between paths
  - Enter paths, eliminating any path whose correlation coefficient with a previously-entered path is > .5
  - $$P(T \le a) = \prod_{i=1}^{\#\text{ remaining paths}} P(p_i \le T)$$

PERT Disadvantages

- Validity of Beta distribution for activity durations
- Validity of central limit theorem for project duration
  - Activity durations are not independent!
- Take into consideration only critical path
  - Not just sum of random variables -- have max. at joins
  - Leads to **overoptimism & underestimation of duration**
- Multiple time estimates required to calibrate
  - Can be time consuming

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  - General
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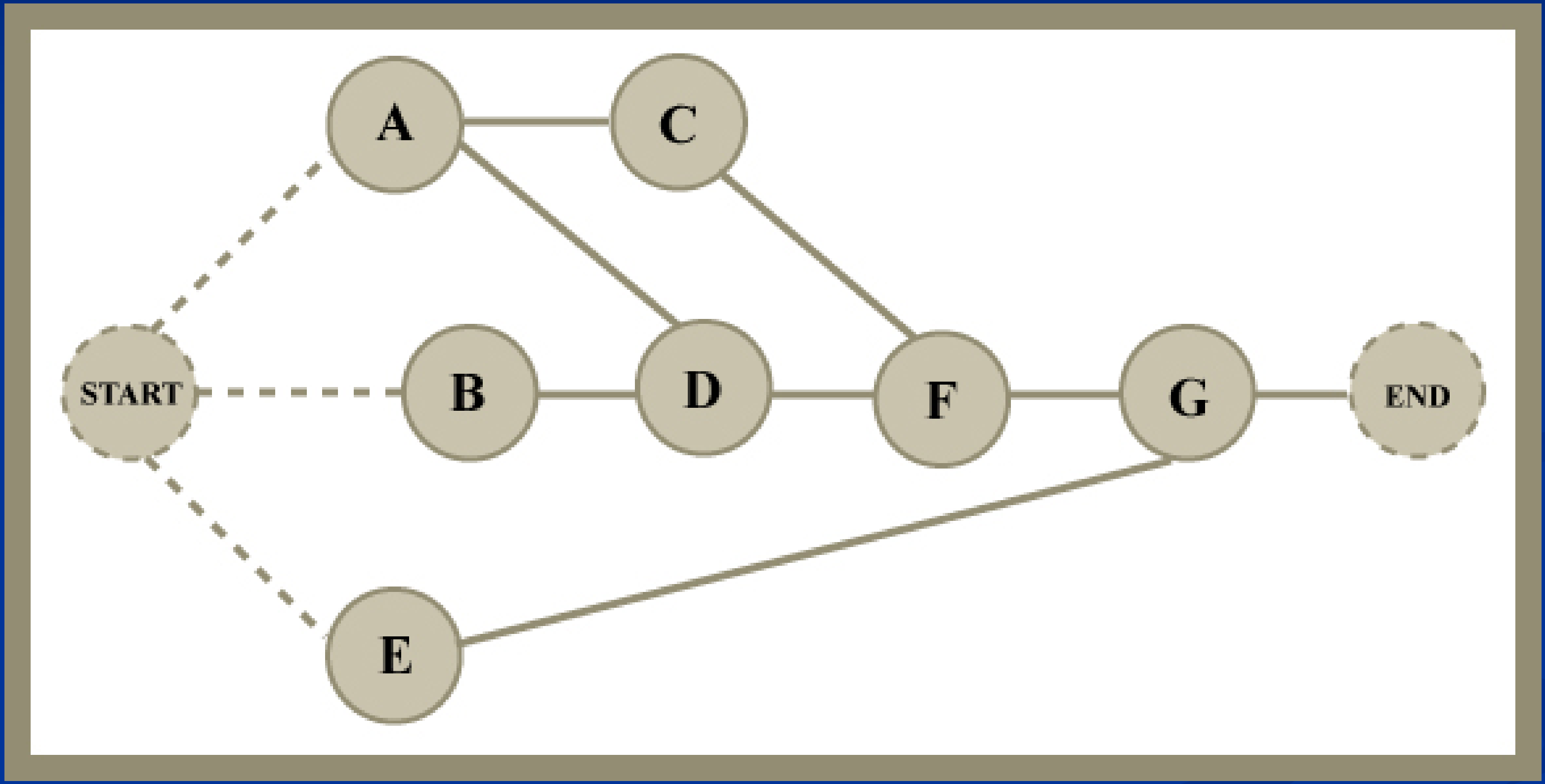
Monte Carlo Simulation Characteristics

- Replaces analytic solution with raw computing power
  - Avoids need to simplify to get analytic solution
  - No need to assume functional form of activity/project distributions
- Used by Van Slyke (1963)
- Aimed at solving the merge bias problem in PERT
- Allows determining the *criticality index* of an activity (Proportion of runs in which the activity was in the critical path)
- Hundreds to thousands of simulations needed

Monte Carlo Simulation Process

- Set the duration distribution for each activity
  - No functional form of distribution assumed
  - Could be joint distribution for multiple activities
- Iterate: for each “trial” (“realization”)
  - Sample random duration from each distributions
  - Find critical path & durations with *standard CPM*
    - Record these results
- Report recorded results
  - Duration distribution
  - Per-node criticality index (% runs where critical)

Network



Monte Carlo Simulation Example

Statistics for Example Activities

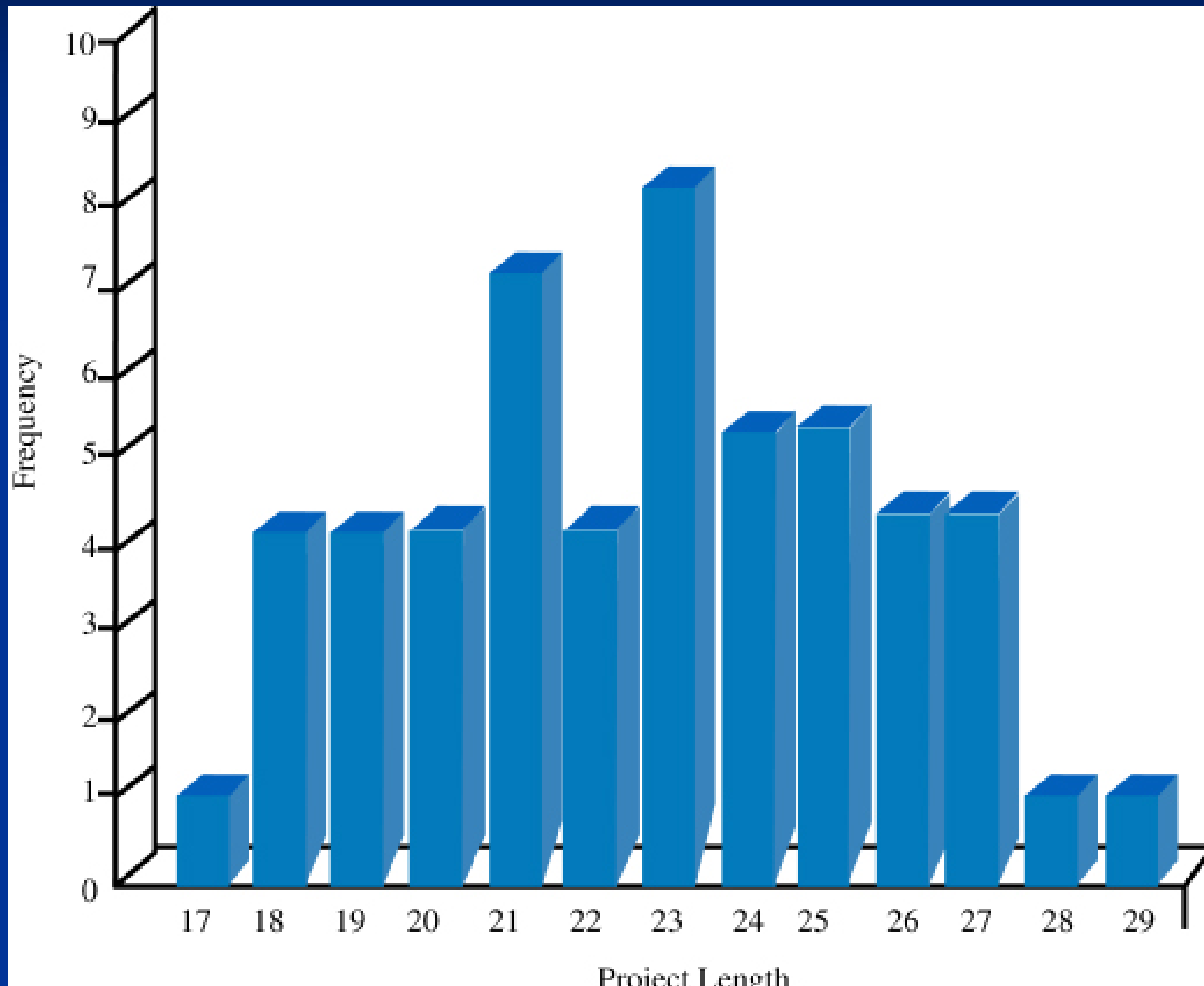
Activity	Optimistic Time, a	Most Likely Time, m	Pessimistic Time, b	Expected Value, d	Standard Deviation , s
A	2	5	8	5	1
B	1	3	5	3	0.66
C	7	8	9	8	0.33
D	4	7	10	7	1
E	6	7	8	7	0.33
F	2	4	6	4	0.66
G	4	5	6	5	0.33

Monte Carlo Simulation Example

Summary of Simulation Runs for Example Project

	Activity Duration								
Run Number	A	B	C	D	E	F	G	Critical Path	Completion Time
1	6.3	2.2	8.8	6.6	7.6	5.7	4.6	A-C-F-G	25.4
2	2.1	1.8	7.4	8.0	6.6	2.7	4.6	A-D-F-G	17.4
3	7.8	4.9	8.8	7.0	6.7	5.0	4.9	A-C-F-G	26.5
4	5.3	2.3	8.9	9.5	6.2	4.8	5.4	A-D-F-G	25.0
5	4.5	2.6	7.6	7.2	7.2	5.3	5.6	A-C-F-G	23.0
6	7.1	0.4	7.2	5.8	6.1	2.8	5.2	A-C-F-G	22.3
7	5.2	4.7	8.9	6.6	7.3	4.6	5.5	A-C-F-G	24.2
8	6.2	4.4	8.9	4.0	6.7	3.0	4.0	A-C-F-G	22.1
9	2.7	1.1	7.4	5.9	7.9	2.9	5.9	A-C-F-G	18.9
10	4.0	3.6	8.3	4.3	7.1	3.1	4.3	A-C-F-G	19.7

Project Duration Distribution



Probability

$$P(X \le \tau) = \frac{\text{Number of Times Project Finished in Less Than or Equal to t weeks}}{\text{Total Number of Replications}}$$

The Probability that the project ends in 20 weeks or less is

$$P(X \le 20) = 13/50 = 26\%$$

Criticality Index

- Definition: Proportion of runs in which the activity was in the critical path
- PERT, CPM assume binary (either 100% or 0%)
- Helpful for prioritizing effort in
  - Monitoring
  - Controlling

How Many Runs are Needed? Criticality Index p (particular node)

- Originally very conservative (10K runs)
- Empirical tests suggest $\leq$ 1000 runs adequate
- Estimate of confidence interval for criticality
  - (1- α) confidence interval=symmetric interval around $\hat{p}$ such that P(true value *p* is within that interval) is (1-α)%

$$\left(\hat{p} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

  - Consider a 95% confidence interval with 10%$\leq$ p $\leq$ 90%, 400$\leq$n $\leq$1000. Then with 95% confidence, $\hat{p}$ will be within 2%-5% of p

How Many Runs are Needed? Mean Project Duration

- Must make assumptions regarding coefficient of variation = σ/μ   (i.e. Std Dev/Mean)
- Basic formula ±Error % ≈ $100\frac{\sigma Z_{\frac{\alpha}{2}}}{\hat{\mu}\sqrt{n}}$
- For Empirical range of CoV (5%..15%)
  - Sample size 400:   within .5% to 1.5% of true value μ
  - Sample size 1000:   within .3% to 1% of true value μ
- Note inverse-root relationship: Halving error requires increasing # of trials by a factor of 4!

How Many Runs are Needed? Project Duration Standard Deviation

- Basic formula $\pm\text{Error}\ \% \approx \dfrac{100Z_{\frac{\alpha}{2}}}{\sqrt{2n}}$
- Sample size 400: $\hat{\sigma}$ within 7% of true value σ
- Sample size 1000: $\hat{\sigma}$ within 4.38% of true value σ
- Inverse-root relationship again present

Monte Carlo: Summary

- Conceptually simple
  - Standard CPM used
  - No need for special assumptions about functional form of distributions
- Provides criticality index (valuable prioritization)
- Scalable analysis quality (albeit with super-linear effort required to reduce error)
- Computationally expensive
- Estimation of duration distributions can be expensive

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(Dynamic) Simulation Approach

- CPM-Based methods use simple representations
  - One-pass: No iteration
  - Represented uncertainty only with respect to *duration*
- Explicitly representing *process* brings benefits
  - Reasoning about process design
  - Identifying *emergent behavior* (e.g. dynamic bottleneck)
  - Simpler estimation of some uncertainties
- Must be clear about whether representations are just *process*-level or also *project*-level

Detailed Representation

- Repetitive processes for which aggregate representation is not desirable
- Processes where *static* planning is not possible
  - Repetitive processes for which # cycles unknown
  - Scheduling and coordinating complex interactions (Large #s of brief interactions, dependent on timing)
  - Cases where timing uncertainties change schedule
- Cases where individual timing component can be estimated, but where aggregate stats not known

Examples of Repetitive Processes

- Earth moving
- Tunneling
- Hotel/Apartment/Dormitory construction
- Road/Bridge construction
- Plumbing and glazing in high-rise

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Simulation Example: Excavation and Transporting

- Given
  - Front-end loader
    - Output: $o_{\text{front-end loader}}$
    - Instantaneous time between loads
  - Trucks
    - n vehicles
    - Capacity c
    - Load time t_l
    - Instantaneous dump time
    - Fully loaded speed s_l , empty speed s_e
    - Distance to dumpsite d

$$o_{\text{trucks}} = \frac{nc}{\frac{d}{s_l} + \frac{d}{s_e}} = \frac{ncs_l s_e}{d(s_e + s_l)}$$

- Naïve productivity: $\min(o_{\text{front-end loader}}, o_{\text{trucks}})$