# 1.017/1.010 Examples of Combinatorial Probability Derivations

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## Example 1

Suppose we toss a die twice. Then the number of possible outcomes is the product of $n_1$ = 6 outcomes on the first toss and $n_1$ = 6 outcomes on the second toss, or $n_1n_2$ = 36. Now define the event ***A*** to correspond to **exactly** one 3 in 2 tosses. This can occur in two ways. First, we can obtain a 3 on the first toss ($n_1$= 1, because there is only one way to achieve a 3 on the first toss) but not on the second ($n_5$= 5 because there are 5 ways to not obtain a 3 on the second toss), giving $n_1n_2$ = 5. Alternatively, we can obtain a 3 on the second toss but not on the first, giving another $n_1n_2$ = 5 outcomes, for a total of 10 outcomes in ***A***. Compute $P(\boldsymbol{A})$ from:

$$P(A) = \frac{n(A)}{n(S)} = \frac{10}{36} = 0.278$$

The same result can be obtained by enumerating all 36 outcomes of two die tosses (e.g. on a tree) and then counting the number with exactly one 3 (do this to check the above result). This die toss experiment is an example of **sampling with replacement**, because outcomes are drawn from the same set of 6 possibilities on each toss (or sub-experiment).

### Example 2

Suppose you draw 2 letters at random from a set of 4 (say A, B, C, and D). Each successive draw is from a smaller set of possible outcomes (once the A has been drawn it cannot be drawn again and there is one less possible outcome for the second draw). If the **order** of the letters is relevant, it follows from the product rule that the number of possible letter pairs is:

The 12 ordered pairs produced by this experiment are:

{AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC}

More generally, an **ordered** subset of *k* objects drawn from a set of *n* objects is called a **permutation** of *n* objects taken *k* at a time. The number of permutations of *n* objects taken *k* at a time is written $P_{k,n}$ :

$$P_{k,n} = n(n-1)(n-2)...(n-k+1) = \frac{n!}{(n-k)!}$$

This number can be used to count simple events that result from sampling without replacement experiments where sampling order is important. The MATLAB function `factorial(n)` can be used to evaluate $P_{k,n}$ for relatively small $n$.

If the **order** in which objects are drawn in sampling without replacement **is not relevant** the number of possibilities decreases. In our example, if order is ignored in our letter example the pairs AB and BA, AC and CA, AD and DA, BC and CB, BD and DB, CD and DC are equivalent. The number of equivalent pairs obtained when order is ignored is the number of permutations (12) divided by the number of ways the same two letters can be arranged ( $n = n_1 \cdot n_2 = 2 \cdot 1 = 2$). Consequently, the number of equivalent unordered pairs is 12/2 = 6.

More generally, an unordered subset of $k$ objects drawn a set of $n$ objects is called a **combination** of $n$ objects taken $k$ at a time. The number of combinations of $n$ objects taken $k$ at a time is $C_{k,n}$

$$C_{k,n} = P_{k,n} / P_{1,k} = \frac{n!}{k!(n-k)!}$$

The $k!$ in the denominator is the number of ways that the $k$ objects can be arranged among themselves (i.e. $P_{1,k}$). The MATLAB function `nchoosek(n,k)` evaluates $C_{k,n}$.for relatively small $n$.

**Example 3**

Suppose that a box contains:

- 8 red balls
- 3 white balls
- 9 blue balls

3 balls are drawn **without replacement**. The sample space consists of all possible arrangements of three balls that can be drawn from the box. Since order is unimportant this implies that the total number of outcomes in the sample space is the number of combinations of 20 balls taken 3 at a time:

$$\text{Total outcomes} = n(S) = C_{3,20} = 1140$$

Find the probabilities of the following events in the sample space using combinatorial and virtual experiment (Monte Carlo) approaches. The MATLAB function `balls` carries out the virtual experiment.

**a) Probability that all 3 balls are red.**

The number of outcomes in the event $A$ = "all 3 balls are red" is the number of possible ways to draw 3 balls from the supply of 8 red balls. This is the number of combinations of 8 red balls taken 3 at a time:

$$\text{Outcomes in } A = n(A) = C_{3,8} = 56$$

The probability of the event is therefore:

$$P(A) = \frac{C_{3,8}}{C_{3,20}} = \frac{56}{1140} = 0.049$$

**b) Probability that at least 1 ball is white**

For events involving "at least one" specifications it is usually best to begin by considering the complementary event "none". The number of outcomes in the event $A'$ = "no balls are white" is the number of possible arrangements of 17 nonwhite balls taken 3 at a time:

$$\text{Outcomes in } A' = n(A') = C_{3,17} = 680$$

Consequently the probability of this event is:

$$P(A) = \frac{C_{3,17}}{C_{3,20}} = \frac{680}{1140} = 0.596$$

And the probability of the complementary event $A$ = "at least one white ball" is:

$$P(A) = 1 - \frac{680}{1140} = 0.404$$

**c) Probability that 2 balls are red and 1 ball is white**

This is an example of an event $A$ resulting from 2 actions (Action 1 is selection of 2 red balls and Action 2 is selection of 1 white ball). From the product rule the number of combinations resulting from both actions is:

$$\text{Outcomes in } A = n(A) = (C_{2,8})(C_{1,3}) = (28)(3) = 84$$

The probability is therefore:

$$P(A) = \frac{C_{2,8}C_{1,3}}{C_{3,20}} = \frac{84}{1140} = 0.074$$

**d) Probability that 1 ball is red, 1 ball is white, and 1 ball is blue**

This is basically the same problem as c). The event $A$ now consists of 3 actions (Action 1 is the selection of 1 red balls, Action 2 is the selection of 1 white ball, and Action 3 is the selection of 1 blue ball). From the fundamental principle of counting the number of combinations resulting from all three actions is:

$$\text{Outcomes in } A = n(A) = (C_{1,8})(C_{1,3})(C_{1,9}) = (8)(3)(9) = 216$$

The probability of $A$ is therefore:

$$P(A) = \frac{C_{1,8}C_{1,3}C_{1,9}}{C_{3,20}} = \frac{216}{1140} = 0.189$$

**Example 4**

Suppose we draw 4 cards from a well-shuffled deck of 52 cards. What is the probability of obtaining exactly 2 aces? Use combinatorial and virtual experiment methods.

Here the sample space $S$ consists of all 4-card hands that can be constructed from a deck of 52 cards. Since the cards are drawn without replacement and order is not important the total number of outcomes in the sample space is the number of combinations of 52 cards taken 4 at a time:

$$\text{Total outcomes in } S = C_{4,52} = 270{,}725$$

The event $A$ results from two actions – Action 1 is the selection of 2 aces and Action 2 is the selection of 2 cards that are not aces. There are $C_{2,4}$ ways to obtain 2 aces out of 4 in the deck and $C_{2,48}$ ways to obtain 2 non-aces from the remaining 48 cards. Consequently the product rule tells us that the total number of ways to obtain exactly 2 aces is:

$$\text{Outcomes in } A = n(A) = (C_{2,4})(C_{2,48}) = (6)(1128) = 6768$$

The probability is therefore:

$$P(A) = \frac{C_{2,4}C_{2,48}}{C_{4,52}} = \frac{6768}{270{,}725} = 0.025$$

**Example 5**

A shelf has 6 math books and 4 physics books that have been placed there at random. Find the probability that 3 particular math books will be grouped together (adjoining), using both combinatorial and virtual experiment methods.

Here it is convenient to define the sample space $S$ to be all possible arrangements of the math books, **with order considered** (because the books must be adjoining). The number of ordered outcomes in this space is the number of permutations of 10 books arranged in 10 locations:

$$\text{Total outcomes in } S = P_{10,10} = 10! = 3{,}628{,}800$$

Now treat the 3 adjoining math books as one group and each of the other 7 books as a distinct group, giving a total of 8 groups. The number of ways for Event ***A*** to occur is the number of ways that these 8 groups can be arranged in the 8 "spaces" on the shelf. Considering order, this is $P_{8,8.}$ For each of these arrangements there are $P_{3,3}$ ways to arrange the 3 books. Consequently, the total number of ways to obtain the specified event is:

$$\text{Outcomes in } A = n(A) = (P_{3,3})(P_{8,8}) = (3!)(8!) = (6)(40320) = 241{,}920$$

The probability is therefore:

$$P(A) = \frac{P_{3,3}P_{8,8}}{P_{10,10}} = \frac{241{,}920}{3{,}628{,}800} = 0.0667$$

**Example 6**

What is the probability of obtaining exactly 3 "6's" in 5 tosses of a single fair die? Use both combinatorial and virtual experiment methods.

This experiment mixes aspects of sampling with and without replacement. The sample space $S$ consists of $6^5$ possible outcomes from the 5 rolls (sampling with replacement):

$$\text{Total outcomes in } S = 6^5 = 7776$$

As in earlier examples, the event $A$ results from 2 actions, Action 1 being the roll of 3 "6's" and Action 2 being the roll of 2 other numbers (not "6's"). Since order is not important there are ${}_5C_3$ ways to obtain the 3 "6's" (sampling without replacement). There are $5^2$ ways to obtain results other than 6 on the remaining 2 tosses (sampling with replacement). Consequently, the total number of ways to obtain the specified event $A$ is:

$$\text{Outcomes in } A = (C_{3,5})(5^2) = 250$$

The probability is therefore:

$$P(A) = \frac{C_{3,5}(5)^2}{(6)^5} = \frac{250}{7776} = 0.032$$