1.017/1.010 Class 5
Combinatorial Methods for Deriving Probabilities

Deriving Probabilities

The basic idea of the conceptual/deductive approach for deriving probabilities is to break the composite experiment into parts (sub-experiments). These parts are selected so that events for each part have readily identified probabilities (e.g. they are equally likely). The rules of probability can then be used to derive the probability of complex events for the composite experiment.

The approach is usually applied to problems with a finite number of discrete outcomes.

The simplest application is an experiment that divides into a number of sub-experiments with **independent equally likely outcomes**.

In this case, suppose A is the event of interest and S is the sample space.

Then the probability of A is the ratio of the number of outcomes $N(A)$ in A to the total number of outcomes $N(S)$:

$$P(A) = \frac{N(A)}{N(S)}$$

Types of Experiments

To evaluate numbers of outcomes and probabilities we need to distinguish different kinds of experiments:

Sampling with replacement: Observed sub-experiment outcomes can reoccur in subsequent trials
vs
Sampling without replacement: Observed sub-experiment outcomes cannot reoccur in subsequent trials

Sub-experiment outcomes are **ordered**
vs.
Sub-experiment outcomes are not **ordered**

Product Rule

Many combinatorial methods rely on the counting or **product rule**, which relates total experimental outcomes to sub-experiment outcomes:

If an experiment is divided into k successive sub-experiments and sub-experiment i yields n_i outcomes, the total number of possible outcomes for the experiment is the product $n_1 n_2 \ldots n_k$

Permutations and Combinations

When experiments involve selecting ***k* items from a group of *n*** and **order matters** we often need to compute number of **permutations**:

$$P_{k,n} = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

When outcomes involve selecting ***k* items from a group of *n*** and **order does not matter** we often need to compute number of **combinations**:

$$C_{k,n} = P_{k,n} / P_{1,k} = \frac{n!}{k!(n-k)!}$$

Examples

Consider following letters written on 4 cards: $A,\ a,\ B,\ C$.
Experiment consists of 2 successive random draws from S. We wish to know probability of getting an A in one draw and a B in the other.

With replacement (cards are replaced after they are drawn):

If Event = E = {AB, aB} (**order matters in event definition**, B must be obtained on draw 2):

N(S) =(4)(4)=16 ; 4 outcomes possible on each draw, product rule

Consider {A, a} to be a **group with 2 elements**. Acceptable number of outcomes from this group on Draw 1 is $P_{1,2}$ = 2 (either A or a). Acceptable number of ways group outcome can be allocated is $P_{1,1}$ = 1 (Draw 1 only).

N(E) = ($P_{1,2}$)($P_{1,1}$)= (2)(1) = 2

P(E)=1/8

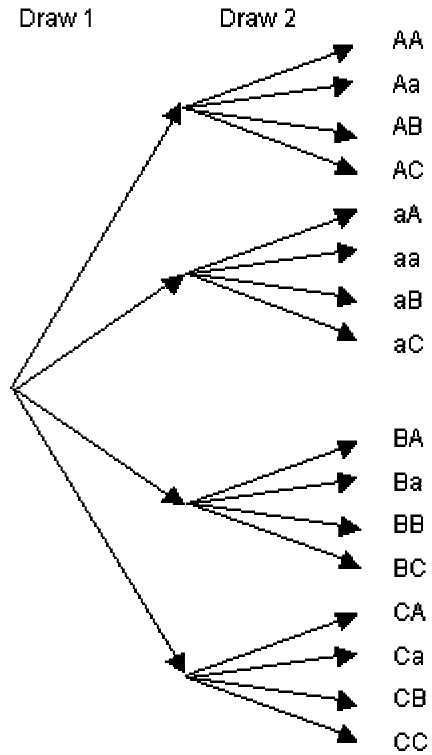
If Event = E = {AB, aB, BA, Ba} = (**order does not matter in event definition**):

N(S) = (4)(4)=16

Acceptable number of ways outcome from group {A, a} can be allocated is $P_{1,2}$ = 2 (Draws 1 or 2)

$N(E) = (P_{1,2})(P_{1,2}) = (2)(2) = 4$

$P(E) = 1/4$



Ordered outcomes for sampling with replacement

Without replacement (cards are not replaced after they are drawn):

If Event = E = {AB, aB} (**order matters in event definition**, B must be obtained on Draw 2):

$N(S) = P_{2,4} = (4)(3) = 12$; 4 outcomes possible on Draw 1, 3 outcomes possible on Draw 2

Acceptable number of outcomes from group {a, A} on Draw 1 is $P_{1,2} = 2$. Acceptable number of ways group outcome can be allocated is $P_{1,1} = 1$ (Draw 1 only)

$N(E) = (P_{1,2})(P_{1,1}) = (2)(1) = 2$

$P(E)=1/6$

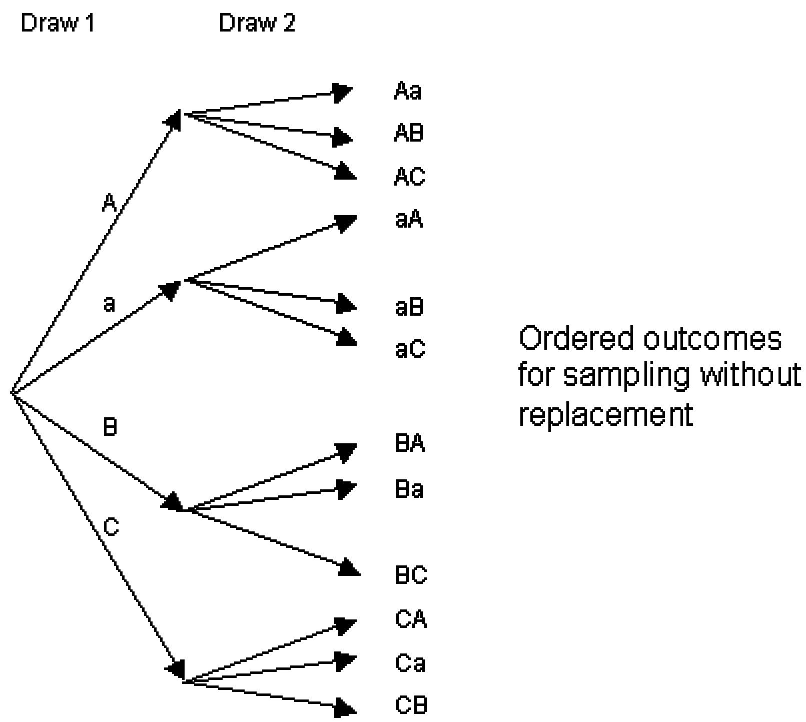
If Event $= E = \{AB, aB, BA, Ba\} =$ (**order does not matter in event defn**):

$N(S) = P_{2,4} = (4)(3) = 12$

Acceptable number of ways outcome from group $\{A, a\}$ can be allocated is $P_{2,2} = 2$ (Draws 1 or 2)

$N(E) = (P_{1,2})(P_{2,2}) = (2)(2) = 4$

$P(E) = 1/3$



When sampling with replacement where order does not matter $N(S)$ and $N(E)$ may be redefined so only combinations rather than permutations are distinguished. Tree can be redrawn accordingly. In above example this gives:

$N(S) = C_{2,4} = 6$
$N(E) = (C_{1,2})(C_{2,2}) = (2)(1) = 2$

$P(E) = 1/3$

Also see combinatorial examples

For complex combinatorial problems it is useful to check results with virtual experiments. See example MATLAB code balls.m.