1.017/1.010 Class 4
Joint Probability, Independence, Repeated Trials

Joint Probabilities and Independence

Joint probability of 2 events A and B defined in the same sample space (probability that outcome lies in A **and** B):

$P(AB) = P(C)$; where event $C = A \cap B = AB$

If A and B are **independent** then:

$P(AB) = P(A)P(B)$

Note that **mutually exclusive events are not independent** since if one occurs we know the other has not.

Example:

Consider the following events A and B defined from a die toss experiment with outcomes $\{1, 2, 3, 4, 5, 6\}$

$A = \{2, 4, 6\}$ $B = \{1, 2, 3, 4\}$

Then:

$P(A) = 1/2$, $P(B) = 2/3$, $P(AB) = 2/6 = P(A)P(B)$

So A and B are independent.

Composite experiments

Related experiments are often conducted in a sequence.

For example, suppose we toss a fair coin (with 2 equally likely outcomes $\{H\ T\}$) and then throw a fair die (with 6 equally likely outcomes $\{1, 2, 3, 4, 5, 6\}$). This process can be viewed as two separate experiments E_1 and E_2 with different sample spaces.

Or … it can be viewed as a single composite experiment E (with 12 ordered equally likely outcomes $\{ H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6 \}$.

Events defined in E_1 and E_2 have equivalent events in E.

Example:

A_2 = { 2, 3 } in E_2 corresponds to A = { $H2, H3, T2, T3$ } in E.

A particular ordered sequence of events from E_1 and E_2 also has an equivalent event in E :

Example:

$A_1 = \{H\}$ in E_1 then $A_2 = \{2, 3\}$ in E_2 corresponds to $A = \{ H2, H3 \}$ in E.

Suppose that A is the composite experiment event that corresponds to event A_1 from experiment E_1 and then event A_2 from experiment in E_2.

A_1 and A_2 are **independent** if:

$$P_E(A) = P_{E1}(A_1)P_{E2}(\mathrm{A}_2)$$

The subscript on each probability identifies the corresponding experiment and sample space.

The events A_1 and A_2 defined in the above coin toss/die roll example satisfy the independence requirement.

Repeated trials

Repeated identical experiments are called **repeated trials**.

Example:

Consider a composite experiment composed of 3 successive fair coin tosses.

This experiment can yield 2^3 = 8 equally likely ordered outcomes:

$\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

The probability of the event A = {exactly 2 heads in 3 tosses} is the fraction of total number of outcomes that yield exactly 2 heads:

$P(A)$ = 3/8

Now consider a particular composite experiment event:

$A_1 = \{H\}$ then $A_2 = \{H\}$ then $A_3 = \{T\}$

Suppose the repeated trials are independent. Then the probability of this composite event is:

$$P(\ A_1 \text{ then } A_2 \text{ then } A_3\ ) = P(A_1)P(A_2)P(\ A_3) = (1/2)(1/2)(1/2) = 1/8.$$

This is one of 3 mutually exclusive repeated trial event sequences that yield exactly 2 heads. It follows that the probability of exactly 2 heads is 3(1/8) = 3/8. Since this is equal to the probability obtained from the composite experiment the independence assumption is confirmed.