1.010 Uncertainty in Engineering
Fall 2008

1.010 - Brief Notes #4
Random Vectors

A set of 2 or more random variables constitutes a random vector. For example, a random vector with two components, $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, is a function from the sample space of an experiment to the (x_1, x_2) plane.

- **<u>Discrete Random Vectors</u>**
  - <u>Characterization</u>
    - *Joint PMF of* X_1 *and* X_2:

      $$P_{\underline{X}}(\underline{x}) = P_{X_1,X_2}(x_1, x_2) = P[(X_1 = x_1) \cap (X_2 = x_2)]$$

    - *Joint CDF of* X_1 *and* X_2:

      $$F_{\underline{X}}(\underline{x}) = F_{X_1,X_2}(x_1, x_2) = P[(X_1 \le x_1) \cap (X_2 \le x_2)]$$

      $$= \sum_{u_1 \le x_1} \sum_{u_2 \le x_2} P_{X_1,X_2}(u_1, u_2)$$

  - <u>Marginal Distribution</u>
    - *Marginal PMF of* X_1:

      $$P_{X_1}(x_1) = P[X_1 = x_1] = \sum_{\text{all } x_2} P[(X_1 = x_1) \cap (X_2 = x_2)] = \sum_{\text{all } x_2} P_{X_1,X_2}(x_1, x_2)$$

    - *Marginal CDF of* X_1:

      $$F_{X_1}(x_1) = P[X_1 \le x_1] = P[(X_1 \le x_1) \cap (X_2 < \infty)] = F_{X_1,X_2}(x_1, \infty) = \sum_{\text{all } x_2} \sum_{u \le x_1} P_{X_1,X_2}(u, x_2)$$

- **Continuous Random Vectors**

  - Characterization

    - *Joint CDF* $F_{X_1,X_2}(x_1, x_2)$: same as for discrete vectors.

    - *Joint Probability Density Function (JPDF) of* $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $f_{X_1,X_2}(x_1, x_2)$:

      This function is defined such that:

      $$f_{X_1,X_2}(x_1, x_2)dx_1dx_2 = P[(x_1 \le X_1 < x_1 + dx_1) \cap (x_2 \le X_2 < x_2 + dx_2)]$$

      Relationships between f_{X_1,X_2} and F_{X_1,X_2} :

      $$f_{X_1,X_2}(x_1, x_2) = \frac{\partial^2 F_{X_1,X_2}(x_1, x_2)}{\partial x_1 \partial x_2}$$

      $$F_{X_1,X_2}(x_1, x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f_{X_1,X_2}(u_1, u_2)du_1\,du_2$$

  - Marginal distribution of X_1

    - *CDF*: $F_{X_1}(x_1) = F_{X_1,X_2}(x_1, \infty)$

    - *PDF*: $f_{X_1}(x_1) = \frac{dF_{X_1}(x_1)}{dx_1} = \frac{\partial F_{X_1,X_2}(x_1, \infty)}{\partial x_1}$

      $$= \frac{\partial}{\partial x_1}\left(\int_{-\infty}^{x_1} du_1 \int_{-\infty}^{\infty} f_{X_1,X_2}(u_1, u_2)du_2\right)$$

      $$= \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, u_2)du_2$$

  - Conditional PDF of $(X_1 \mid X_2 = x_2)$

    $$f_{(X_1|X_2=x_2)}(x_1) = \frac{f_{X_1,X_2}(x_1, x_2)}{f_{X_2}(x_2)}$$

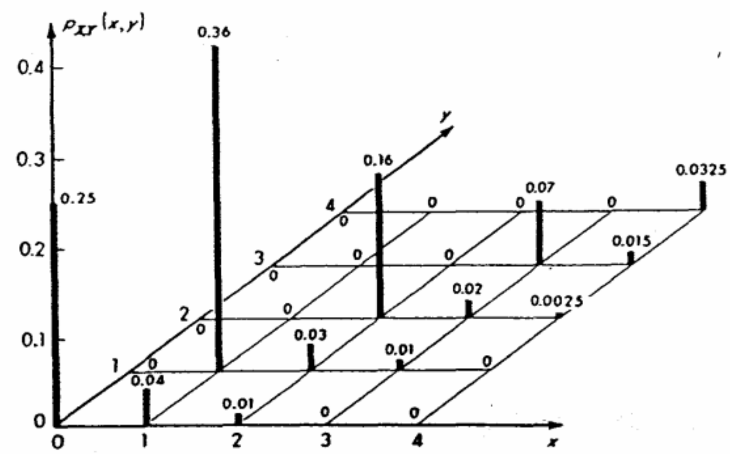
    $$\propto \quad f_{X_1,X_2}(x_1, x_2), \text{ for } f_{X_2}(x_2) \ne 0$$

- <u>Conditional Distribution</u>

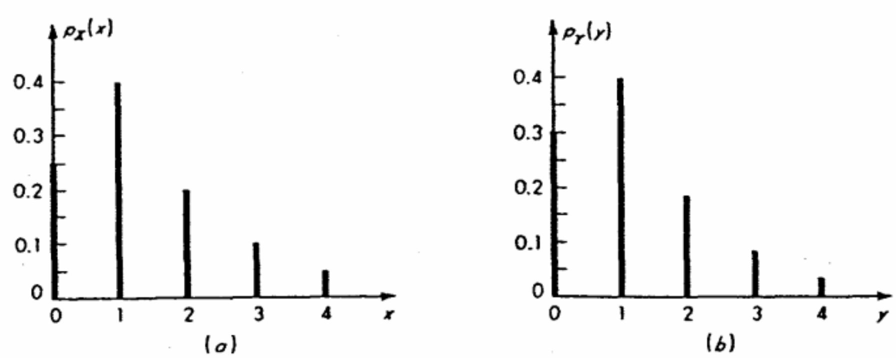
  - *Conditional PMF of* $(X_1 \mid X_2 = x_2)$:

$$P_{(X_1|X_2=x_2)}(x_1) = P[X_1 = x_1 \mid X_2 = x_2] = \frac{P_{X_1,X_2}(x_1, x_2)}{P_{X_2}(x_2)}$$

$$\propto \; P_{X_1,X_2}(x_1, x_2)$$



Example of discrete joint distribution: joint PMF of traffic at remote location
(X in cars/30 sec. interval) and traffic recorded by some imperfect traffic counter (Y)
(note: X and Y are the random variables X_1 and X_2 in our notation).



Example of discrete joint distribution: marginal distributions.
(a) Marginal PMF of actual traffic X, and (b) marginal counter response Y.

- **Independent Random Variables**

X_1 and X_2 are independent variables if:

$$F_{X_1,X_2}(x_1,x_2) = F_{X_1}(x_1) \,.\, F_{X_2}(x_2)$$

Equivalent conditions for continuous random vectors are:

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) \,.\, f_{X_2}(x_2)$$

or:

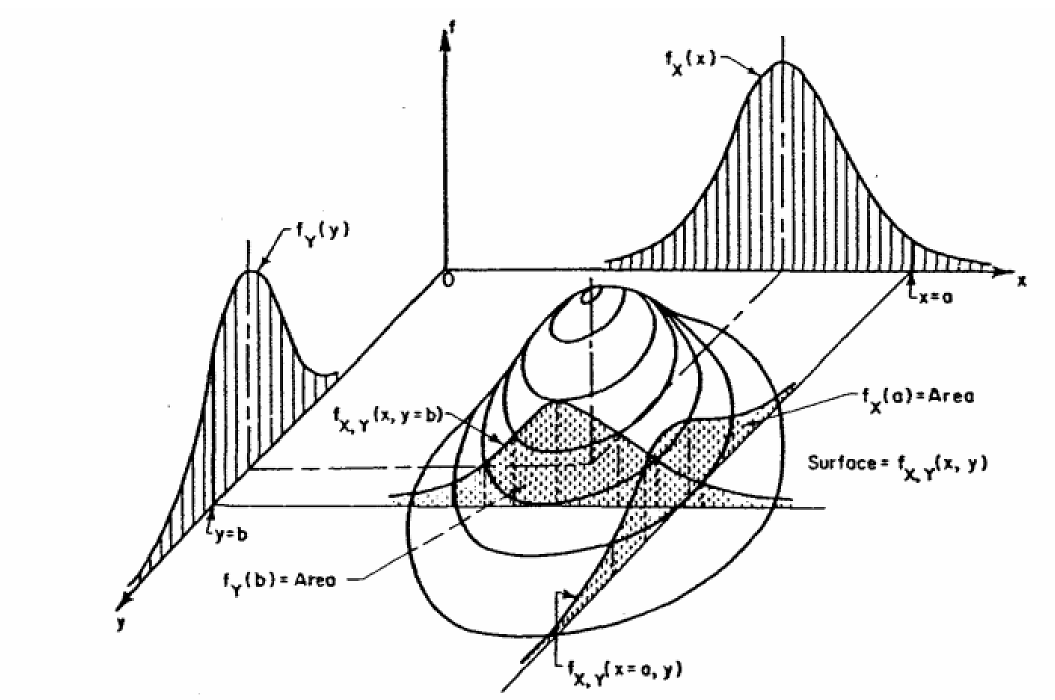
$$f_{(X_1|X_2=x_2)}(x_1) = f_{X_1}(x_1)$$

and for discrete random vectors:

$$P_{X_1,X_2}(x_1,x_2) = P_{X_1}(x_1) \,.\, P_{X_2}(x_2)$$

or:

$$P_{(X_1|X_2=x_2)}(x_1) = P_{X_1}(x_1)$$



Example of continuous joint distribution:
joint and marginal PMF of random variables X and Y.
(Note: X and Y are the random variables X_1 and X_2 in our notation)