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1. Taking the problem's cue to determine the magnitude of discontinuity of the wavefunction at the origin:

$$\langle q|\hat{H}|\psi\rangle = E\langle q|\psi\rangle$$

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} dq\,\langle q|\hat{H}|\psi\rangle = \lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} dq\, E\langle q|\psi\rangle$$

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} dq\Big[\underbrace{\frac{-\hbar^2}{2m}\frac{d^2\psi(q)}{dq^2}}_{\text{Discontinuous function}} + \underbrace{V_0\delta(q)\psi(q)}_{\text{Infinite function}}\Big] = \underbrace{\cancel{\lim_{\epsilon\to 0} E\int_{-\epsilon}^{+\epsilon} dq\,\langle q|\psi\rangle}}_{\text{A vanishing integral over a finite function}} \to 0$$

$$\lim_{\epsilon\to 0} -\frac{\hbar^2}{2m}\left[\int_0^{+\epsilon} dq\,\frac{d^2\psi(q)}{dq^2} - \int_0^{-\epsilon} dq\,\frac{d^2\psi(q)}{dq^2}\right] + \int_{-\epsilon}^{+\epsilon} dq\, V_0\,\delta(q)\psi(q) = 0$$

$$\left.\frac{d\psi_R(q)}{dq}\right|_{q=0} - \left.\frac{d\psi_L(q)}{dq}\right|_{q=0} - \frac{2mV_0}{\hbar^2}\psi(q)\Big|_{q=0} = 0$$

$$\therefore \left.\frac{d\psi_L(q)}{dq}\right|_{q=0} - \left.\frac{d\psi_R(q)}{dq}\right|_{q=0} = \frac{-2mV_0}{\hbar^2}\psi(0)$$

Physically: The particle is moving freely until it passes $q=0$, at which point the delta function potential effectively "kicks" the particle, imparting a discontinuous change in its momentum. This can be seen by noting that $\hat{p} \leftrightarrow -i\hbar\frac{d}{dq}$

and thus: $$\underbrace{\langle q|\hat{p}|\psi_L\rangle\big|_{q=0} - \langle q|\hat{p}|\psi_R\rangle\big|_{q=0}}_{\text{Change in momentum upon crossing } q=0} = \underbrace{\frac{2imV_0}{\hbar}\psi(0)}_{\text{Non zero if } V_0\neq 0}$$

2. a. We have for the wavefunctions $|\psi_L\rangle = A_L|+p\rangle + B_L|-p\rangle$

$|\psi_R\rangle = A_R|+p\rangle + B_R|-p\rangle$

In free space, the solution is plane waves, so we can (without loss of generality) use this basis to express the solution.

$$\langle q|+p\rangle \propto e^{ipq/\hbar};\ \langle q|-p\rangle \propto e^{-ipq/\hbar}$$

We have continuity at $q=0$: $\langle q|\psi_L\rangle = \langle q|\psi_R\rangle$

$$A_L\langle q|+p\rangle + B_L\langle q|-p\rangle = A_R\langle q|+p\rangle + B_R\langle q|-p\rangle$$

$$A_L + B_L = A_R + B_R \quad\text{or}\quad A_L + B_L - A_R - B_R = 0 \quad (1)$$

And discontinuity of ψ': $\langle q|\hat{p}|\psi_L\rangle\big|_{q=0} - \langle q|\hat{p}|\psi_R\rangle\big|_{q=0} = \frac{2imV_0}{\hbar}\psi(0)$

$$(A_Lp - B_Lp) - (A_Rp - B_Rp) = \frac{2imV_0}{\hbar}(A_L + B_L)$$

$$A_L - B_L - A_R + B_R = \frac{2imV_0}{p\hbar}(A_L + B_L) \quad (2)$$

(1) and (2) form the boundary conditions imposed by the origin. Since this problem involves particles incident from the right, $|B_R| > 0$, since this is the amplitude moving from the right towards $q=0$. These particles may reflect (A_R) or transmit (B_L) but no physically reasonable mechanism explains particles moving towards $q=0$ from the left. Therefore, $A_L = 0$ (3) and this is the final boundary condition.

2. b. Transmission: $T \equiv \frac{|B_L|^2}{|B_R|^2}$, Reflection: $R \equiv \frac{|A_R|^2}{|B_R|^2}$, Conservation: $T+R=1$

Conditions (2) and (3) give: $B_L = \left(1 - \frac{2imV_0}{p\hbar}\right) B_R - \left(1 + \frac{2imV_0}{p\hbar}\right) A_R$

Apply (1) $: B_R + A_R = \left(1 - \frac{2imV_0}{p\hbar}\right) B_R - \left(1 + \frac{2imV_0}{p\hbar}\right) A_R$

$$A_R = \frac{-\frac{2imV_0}{p\hbar}}{2 + \frac{2imV_0}{p\hbar}} B_R \Longrightarrow R = \frac{\left|-\frac{imV_0}{p\hbar}\right|^2}{\left|1 + \frac{imV_0}{p\hbar}\right|^2} = \frac{1}{1 + \frac{p^2\hbar^2}{mV_0^2}}$$

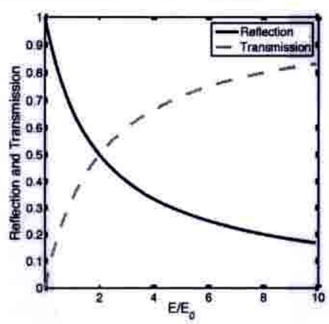
$$B_L = \left(1 - \frac{2imV_0}{p\hbar}\right) B_R + \frac{\left(1 + \frac{2imV_0}{p\hbar}\right)\left(\frac{2imV_0}{p\hbar}\right)}{2 + \frac{2imV_0}{p\hbar}} B_R$$

$$= \frac{\left(1 - \frac{2imV_0}{p\hbar}\right)\left(2 + \frac{2imV_0}{p\hbar}\right) + \left(1 + \frac{2imV_0}{p\hbar}\right)\left(\frac{2imV_0}{p\hbar}\right)}{2 + \frac{2imV_0}{p\hbar}} B_R$$

$$B_L = \frac{2}{2 + \frac{2imV_0}{p\hbar}} B_R \Longrightarrow T = \frac{|2|^2}{\left|2 + \frac{2imV_0}{p\hbar}\right|^2} = \frac{1}{1 + \frac{m^2V_0^2}{p^2\hbar^2}}$$

If we recall that $p = \sqrt{2mE}$; $R = \frac{1}{1 + \frac{2E}{E_0}}$ and $T = \frac{1}{1 + \frac{E_0}{2E}}$
and define $E_0 = \frac{mV_0^2}{\hbar^2}$

A good measure of the barrier is $\frac{E}{E_0}$. For the weak barrier/fast particle limit, $\frac{E}{E_0} \gg 1$ and $T \simeq 1$, $R \simeq 0$. For the strong barrier/slow particle limit, $\frac{E}{E_0} \ll 1$, $T \approx 0$, $R \approx 1$



3. a. For an attractive delta function potential, $V_0 < 1$. The main qualitative difference is that a bound state exists in this potential for energy $E < 0$. In consideration of R and T, V_0 only appears as V_0^2, which is insensitive to the sign. The exact same conditions as above apply with $\frac{E}{E_0}$ indicating the strength of the potential.

b. For $E = -\frac{1}{2}, E_0 = 1$, $R = \frac{1}{1-1} = \infty$, $T = \frac{1}{1-1} = \infty$.

You can solve for the bound state and find $E = -\frac{mV_0^2}{2\hbar^2} = -\frac{E_0}{2} = -\frac{1}{2}$ when $E_0 = 1$

Thus, when the particle has the exact amount of energy to be bound, it won't scatter through or reflect. It's notable that this shows up even with incorrect assumptions much earlier for $E < 0$.