MASSACHUSETTS INSTITUTE OF TECHNOLOGY

5.73 Quantum Mechanics I
Fall, 2002

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Problem Set #2

DUE: At the start of Lecture on Friday, September 20.

Reading: Merzbacher Handout, pp. 92-112.

Problems:

1. $\psi_1(x) = \dfrac{a^2}{b^2(x-x_0)^2 + c^2}$ a, b, and c are real

   A. Normalize $\psi_1(x)$ in the sense $\int_{-\infty}^{\infty} |\psi|^2\, dx = 1$.

   B. Compute values for $\langle x \rangle, \langle x^2 \rangle$, and Δx for $\psi_1(x)$.

   C. (*optional*) Compute values for $\langle k \rangle$ and Δk for $\overline{\psi}_1(k)$, where $\overline{\psi}_1(k)$ is the Fourier transform of $\psi_1(x)$.
   [If you choose not to do this problem, state what you expect for the form of $\Psi_1(k)$ and the magnitude of Δk.]

2. $\psi_2(x) = e^{-c^2(x-b)^2} e^{i\alpha(x)}$ where c, b, and $\alpha(x)$ are real. Use the stationary phase idea to design $\alpha(x)$ in the region of x near x = b so that $\langle k \rangle = k_0 \neq 0$.

3. Merzbacher, page 111, #2.

4. The following problem is one of my "patented" magical mystery tours. It is a very long problem which absolutely demands the use of a computer for parts F and G. There are many separate computer programs that you will need to write for this problem. I urge you to divide the labor into smaller groups, each responsible for a different piece of programming. I believe that the insights you will obtain from working together on this problem will be more than worth the effort expended.

   Consider the simplest possible symmetric double minimum potential:

   $V(x) = a\delta(x) \quad a > 0 \qquad -L/2 < x < L/2$
   $V(x) = \infty \qquad |x| \geq L/2.$

A. Solve for all of the eigenstates and eigen-energies for states that have odd reflection symmetry about $x = 0$. (This part of the problem is very easy.)

B. Solve for the energy eigenstates and eigen-energies for the 5 lowest energy even-symmetry states. Choose a $= 400\hbar^2/\mathrm{Lm}$. I suggest you use trial functions of form

$$\psi_n(x) = N \sin[k_n(x + L/2)] \qquad -L/2 \le x < 0$$
$$\psi_n(x) = -N \sin[k_n(x - L/2)] \qquad 0 < x \le L/2$$

One way to find the eigen-energies is to plot the quantities $y = \tan(kL/2)$ and $y = -kL/400$ and to determine eigen-energies from the k-values at intersections. Each E_n (odd n, even symmetry) is located at an intersection. Note there will be exactly one value of E_n below the lowest odd-symmetry eigenstate (E_2) and one value of E_n between each consecutive pair of odd-symmetry eigenstates.

C. For an ordinary infinite square well. the ratio of the spacing between the two lowest levels to that between the two lowest odd–symmetry levels, is

$$R_{21;42} \equiv \frac{E_2 - E_1}{E_4 - E_2} = \frac{4-1}{16-4} = \frac{3}{12} = 0.25.$$

For your double minimum potential, this level spacing ratio will decrease from 0.25 at a = 0 toward 0 as a increases. For the value of a that I suggested, this ratio should be about 0.003.

Repeat the calculation of $R_{21;42}$ for E_1 using a-values a factor of 3 and 9 smaller than the one you decided on above.

Suggest a functional relationship between a and $R_{21;42}$.

D. The ratio

$$R_{43;42} = 7/12$$

for an ordinary infinite square well. Is the E_4-E_3 spacing you obtained for $a = 400\hbar^2/\mathrm{Lm}$ larger or smaller than E_2–E_1? Why?

E. For $a = 400\hbar^2/LM$, plot

$$\psi_+(x) \equiv 2^{-1/2}(\psi_1 + \psi_2)$$

and

$$\psi_-(x) \equiv 2^{-1/2}(\psi_1 - \psi_2).$$

What does this suggest about the possibility of creating a state localized on the left or right side of the well?

F. Construct $\Psi_+(x,t)$ and $\Psi_-(x,t)$ and compute the following three quantities:

(i). Survival Probability of $\Psi_+(x,0)$

$$P_+(t) = \left| \int \Psi_+^*(x,t)\Psi_+(x,0)dx \right|^2$$

(ii). Survival Probability of $\Psi_-(x,0)$

$$P_\pm(t) = \left| \int \Psi_-^*(x,t)\Psi_-(x,0)dx \right|^2$$

(iii). $\Psi_+(x,t) \rightarrow \Psi_-(x,0)$ Transfer Probability

$$P_{+-}(t) = \left| \int \Psi_+^*(x,t)\Psi_-(x,0)dx \right|^2 .$$

G. Now construct a more elaborate wavepacket from

$$\Psi_L(x,0) = 6^{-1/2}\left[\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\right].$$

There are two critical times in the evolution of $\Psi_L(x,t)$. These are t_g, the global recurrence time for the even-n levels of the infinite box without the δ–function barrier,

$$t_g = \frac{2mL^2}{h}$$

and t_t, the tunneling round trip time for the simple superposition state in part E,

$$t_t = \frac{h}{(E_2 - E_1)}.$$

(i) Plot $|\Psi_L(x,0)|^2$, $\left|\Psi_L\left(x,\frac{8mL^2}{h}\right)\right|^2$, and $\left|\Psi_L\left(x,\frac{h}{(E_2-E_1)}\right)\right|^2$.

Comment on what you see in these 3 plots. There is a huge amount of information. "Assign" as many features or families of features as you can.

(ii) Calculate the following quantities and plot the following quantities twice, once over a short $0 \le t \le 2t_g$ and once over a long $0 \le t \le t_t$ time interval,

$$\langle x \rangle_t = \int \Psi_L^*(x,t) x \Psi_L(x,t) dx$$

$$\langle x^2 \rangle_t = \int \Psi_L^*(x,t) x^2 \Psi_L(x,t) dx$$

$$\Delta x_t = \left[\langle x^2 \rangle_t - \langle x \rangle_t^2\right]^{1/2}.$$

(iii) Compare $\langle x \rangle_t$ and Δx_t and explain why the position variance exhibits periodic crashes toward 0. What might account for such a focussing of the wavepacket?