Angular Momentum

Since $\hat{\mathbf{L}}^2$ and $\hat{L}_z$ commute, they share common eigenfunctions. These functions are extremely important for the description of angular momentum problems – they determine the allowed values of angular momentum and, for systems like the Rigid Rotor, the energies available to the system. The first things we would like to know are the eigenvalues associated with these eigenfunctions. We will denote the eigenvalues of $\hat{\mathbf{L}}^2$ and $\hat{L}_z$ by α and β, respectively so that:

$$\hat{\mathbf{L}}^2 Y_\alpha^\beta(\theta,\phi) = \alpha Y_\alpha^\beta(\theta,\phi) \qquad \hat{L}_z Y_\alpha^\beta(\theta,\phi) = \beta Y_\alpha^\beta(\theta,\phi)$$

For brevity, in what follows we will omit the dependence of the eigenstates on θ and ϕ so that the above equations become

$$\hat{\mathbf{L}}^2 Y_\alpha^\beta = \alpha Y_\alpha^\beta \qquad \hat{L}_z Y_\alpha^\beta = \beta Y_\alpha^\beta$$

It is convenient to define the raising and lowering operators (note the similarity to the Harmonic oscillator!):

$$\hat{L}_\pm \equiv \hat{L}_x \pm i\hat{L}_y$$

Which satisfy the commutation relations:

$$\left[\hat{L}_+,\hat{L}_-\right] = 2\hbar\hat{L}_z \qquad \left[\hat{L}_z,\hat{L}_\pm\right] = \pm\hbar\hat{L}_\pm \qquad \left[\hat{L}_\pm,\hat{\mathbf{L}}^2\right] = 0$$

These relations are relatively easy to prove using the commutation relations we've already derived:

$$\left[\hat{L}_x,\hat{L}_y\right] = i\hbar\hat{L}_z \qquad \left[\hat{L}_y,\hat{L}_z\right] = i\hbar\hat{L}_x \qquad \left[\hat{L}_z,\hat{L}_x\right] = i\hbar\hat{L}_y \qquad \left[\hat{\mathbf{L}}^2,\hat{L}_z\right] = 0$$

For example:

$$\begin{aligned}\left[\hat{L}_z,\hat{L}_\pm\right] &= \left[\hat{L}_z,\hat{L}_x\right] \pm i\left[\hat{L}_z,\hat{L}_y\right] \\ &= i\hbar L_y \pm i\left(-i\hbar L_x\right) = \pm\hbar\left(L_x \pm iL_y\right) \\ &= \pm\hbar\hat{L}_\pm\end{aligned}$$

The raising and lowering operators have a peculiar effect on the eigenvalue of $\hat{L}_z$:

$$\hat{L}_z(\hat{L}_\pm Y_\alpha^\beta) = (\left[\hat{L}_z,\hat{L}_\pm\right] + \hat{L}_\pm\hat{L}_z)Y_\alpha^\beta = (\pm\hbar\hat{L}_\pm + \hat{L}_\pm\beta)Y_\alpha^\beta = \left(\beta \pm \hbar\right)(\hat{L}_\pm Y_\alpha^\beta)$$

Thus, $\hat{L}_+$ ($\hat{L}_-$) raises (lowers) the eigenvalue of $\hat{L}_z$ by $\hbar$, hence the names. Since the raising and lowering operators commute with $\hat{\mathbf{L}}^2$ they do not change the value of α and so we can write

$$\hat{L}_\pm Y_\alpha^\beta \propto Y_\alpha^{\beta\pm\hbar}$$

and so the eigenvalues of $\hat{L}_z$ are evenly spaced!

What are the limits on this ladder of eigenvalues? Recall that for the harmonic oscillator, we found that there was a minimum eigenvalue and the eigenstates could be created by successive applications of the raising operator to the lowest state. There is also a minimum eigenvalue in this case. To see this, note that

$$\left\langle \hat{L}_x^2+\hat{L}_y^2\right\rangle=\left\langle \hat{L}_x^2\right\rangle+\left\langle \hat{L}_y^2\right\rangle\geq 0$$

This result simply reflects the fact that if you take any observable operator and square it, you **must** get back a positive number. To get a negative value for the average value of $\hat{L}_x^2$ or $\hat{L}_y^2$ would imply an imaginary eigenvalue of $\hat{L}_x$ or $\hat{L}_y$, which is impossible since these operators are Hermitian. Besides, what would an imaginary angular momentum mean? We now apply the above equation for the specific wavefunction Y_α^β:

$$\begin{aligned}0\leq\int Y_\alpha^{\beta*}\left(\hat{L}_x^2+\hat{L}_y^2\right)Y_\alpha^\beta&=\int Y_\alpha^{\beta*}\left(\hat{\mathbf{L}}^2-\hat{L}_z^2\right)Y_\alpha^\beta\\&=\int Y_\alpha^{\beta*}\left(\alpha-\beta^2\right)Y_\alpha^\beta\\&=\alpha-\beta^2\end{aligned}$$

Hence $\beta^2\leq\alpha$ and therefore $-\sqrt{\alpha}\leq\beta\leq\sqrt{\alpha}$. Which means that there are both *maximum* and *minimum* values that β can take on for a given α. If we denote these values by β_{max} and β_{min}, respectively, then it is clear that

$$\hat{L}_+Y_\alpha^{\beta_{\max}}=0\qquad \hat{L}_-Y_\alpha^{\beta_{\min}}=0.$$

We can then use this knowledge and some algebra tricks trick to determine the relationship between α and β_{max} (or β_{min}). First note that:

$$\Rightarrow \hat{L}_-\hat{L}_+Y_\alpha^{\beta_{\max}}=0\qquad \hat{L}_+\hat{L}_-Y_\alpha^{\beta_{\min}}=0$$

We can expand this explicitly in terms of $\hat{L}_x$ and $\hat{L}_x$:

$$\Rightarrow\left(\hat{L}_x^2+\hat{L}_y^2-i(\hat{L}_y\hat{L}_x-\hat{L}_x\hat{L}_y)\right)Y_\alpha^{\beta_{\max}}=0\qquad\left(\hat{L}_x^2+\hat{L}_y^2+i(\hat{L}_y\hat{L}_x-\hat{L}_x\hat{L}_y)\right)Y_\alpha^{\beta_{\min}}=0$$

However, this is not the most convenient form for the operators, because we don't know what $\hat{L}_x$ or $\hat{L}_y$ gives when acting on Y_α^β. However, we can re-write the same expression in terms of $\hat{\mathbf{L}}^2$ and $\hat{L}_z$:

$$\Big(\underbrace{\hat{L}_x^2+\hat{L}_y^2}_{\hat{\mathbf{L}}^2-\hat{L}_z^2}\pm i\underbrace{(\hat{L}_y\hat{L}_x-\hat{L}_x\hat{L}_y)}_{-i\hbar\hat{L}_z}\Big)$$

So then we have

$$\Rightarrow\left(\hat{\mathbf{L}}^2-\hat{L}_z^2-\hbar\hat{L}_z\right)Y_\alpha^{\beta_{\max}}=0\qquad\left(\hat{\mathbf{L}}^2-\hat{L}_z^2+\hbar\hat{L}_z\right)Y_\alpha^{\beta_{\min}}=0$$

$$\Rightarrow\left(\alpha-\beta_{\max}^2-\hbar\beta_{\max}\right)=0\qquad\left(\alpha-\beta_{\min}^2+\hbar\beta_{\min}\right)=0$$

$$\Rightarrow\alpha=\beta_{\max}(\beta_{\max}+\hbar)=\beta_{\min}(\beta_{\min}-\hbar)$$

$$\Rightarrow\beta_{\max}=-\beta_{\min}\equiv\hbar l$$

where in the last line we have simply defined a new variable, l, that is dimensionless (notice that $\hbar$ has the units of angular momentum). So combining these minimum and maximum values we have that $-\hbar l \le \beta \le \hbar l$. Further, since we can get from the lowest to the highest eigenvalue in increments of $\hbar$ by successive applications of the raising operator, it is clear that the difference between the highest and lowest values [$\hbar j - (-\hbar j) = 2\hbar l$] must be an integer multiple of $\hbar$. Thus, l itself must either be an integer or a half-integer.

Putting all these facts together, we conclude (Define $m \equiv \beta / \hbar$):

$$\hat{\mathbf{L}}^2 Y_l^m = \hbar^2 l(l+1) Y_l^m \qquad l = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2 \ldots$$

and

$$\hat{L}_z Y_l^m = m\hbar Y_l^m \qquad m = -l, -l+1 \ldots l-1, l$$

where we have replaced α with l and β with m so that Y_α^β becomes Y_l^m. Also, in the first equation, we have noted that $0 \le \langle \hat{\mathbf{L}}^2 \rangle = \hbar^2 l(l+1)$ implies $l \ge 0$. These are the fundamental eigenvalue equations for all forms of angular momentum.

Notice that there is a difference here from what we saw for the rigid rotor. There, we had:

$$E_J = \frac{\hbar^2}{2I} J(J+1) \qquad J = 0,\ 1,\ 2, \ldots$$

where, as a reminder, the quantum number J for the rigid rotor is equivalent to the quantum number l define above. Here, the dependence of the energy on J – $E \propto J(J+1)$ - is the same as we found in our derivation for $\hat{\mathbf{L}}^2$. The factor of $1/2I$ simply arises from the fact that the rigid rotor Hamiltonian is $\hat{\mathbf{L}}^2/2I$ rather than $\hat{\mathbf{L}}^2$. The real difference is that **half integer values of J do not appear for the rigid rotor**. At first, you might think this means we made a mistake in our derivation above and that l should only be an integer and not a half integer. However, there is no error. The difference arises because our derivation above is valid for **any kind of angular momentum**. Thus, while certain values of l may not appear for certain types of angular momentum (e.g. they don't occur for the rigid rotor) we will see later on that they can appear for other types of angular momentum. Most notably, electrons have an intrinsic spin angular momentum with $l = \frac{1}{2}$. Thus, while individual systems may have additional restrictions on the allowed values of l, angular momentum states always obey the above eigenvalue relations.