

Review Session #2 Sample Problem

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Again, please forgive any typos. This is slightly modified from what was presented in the review session so that there would be only one root (I glossed over the fact that the square root implied there were two roots, positive and negative). As a result, things may be off by a factor of m or g here and there, but the important things to take away from this sample problem are what we are asking you to do and how you would go about doing it.

We imagine we have a spring, suspended from the ceiling, and at the end of it is attached a block with mass m . This is a special spring, so instead of the normal spring force equation, its force is defined proportional to the cube of its extension.

So, the force balance equation that we have is

$$F = mg - kx^3 - fv$$

Where g is acceleration due to gravity, k is the spring constant, f is the frictional constant, and v is the velocity of the block.

We start the block and spring completely at rest (meaning the spring is neither extended nor compressed) and would like to analyze the steady state of our system. The system above can be stated as:

$$m \frac{d^2x}{dt^2} = mg - kx^3 - f \frac{dx}{dt} \quad \text{or just} \quad \frac{d^2x}{dt^2} = g - \frac{k}{m} x^3 - \frac{f}{m} \frac{dx}{dt}$$

The first question is: how would we pose this as a problem that could be easily solved numerically by Matlab?

We need to make this second-order ODE into a system of first-order ODEs that Matlab can handle easily. We do this by defining the intermediate variable \underline{y} , where

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x \\ \frac{dx}{dt} \end{bmatrix} \quad \text{and thus} \quad \frac{d\underline{y}}{dt} = \begin{bmatrix} y_2 \\ g - \frac{k}{m} y_1^3 - \frac{f}{m} y_2 \end{bmatrix}$$

So we now have the equation we need to give to Matlab.

The second question is: what is the steady state of the system?

We do that by setting the time derivative of \underline{y} to zero. This makes sense, because at our steady state, we would see not only that the block is not moving (so $dy_1/dt = 0$), but also that there is no net force on the block (so $dy_2/dt = 0$). This gives us:

$$y_2 = 0$$

$$g - \frac{k}{m} y_1^3 - \frac{f}{m} y_2 = 0$$

Plugging the equation for y_2 into the second equation, we immediately see that

$$y_1 = \sqrt[3]{\frac{gm}{k}}$$

So our steady state in \underline{y} is $\underline{y}_{ss} = \begin{bmatrix} \sqrt[3]{\frac{gm}{k}} \\ 0 \end{bmatrix}$.

The next question is: is this steady state stable?

To find this out, we need to evaluate the Jacobian at the steady state and find the eigenvalues of the resulting matrix. The Jacobian defined by the equations in \underline{y} is:

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{3k}{m} y_1^2 & -\frac{f}{m} \end{bmatrix} \quad \text{so} \quad J|_{\underline{y}_{ss}} = \begin{bmatrix} 0 & 1 \\ -\frac{3k}{m} y_1^2 & -\frac{f}{m} \end{bmatrix} \bigg|_{\underline{y}_{ss}} = \begin{bmatrix} 0 & 1 \\ -\frac{3k}{m} \left(\frac{gm}{k}\right)^{\frac{2}{3}} & -\frac{f}{m} \end{bmatrix}$$

To find the eigenvalues of that steady state, we need to solve the equation

$$|J - \lambda I| = 0, \quad \text{so} \quad \begin{vmatrix} -\lambda & 1 \\ -\frac{3k}{m} \left(\frac{gm}{k}\right)^{\frac{2}{3}} & -\frac{f}{m} - \lambda \end{vmatrix} = 0 = \lambda^2 + \frac{f}{m} \lambda + \frac{3k}{m} \left(\frac{gm}{k}\right)^{\frac{2}{3}}$$

This is a simple quadratic, where the solution is

$$\lambda = \frac{-\frac{f}{m} \pm \sqrt{\frac{f^2}{m^2} - \frac{12k}{m} \left(\frac{gm}{k}\right)^{\frac{2}{3}}}}{2} = \frac{-f \pm \sqrt{f^2 - 12k^{\frac{1}{3}} m^{\frac{5}{3}} g^{\frac{2}{3}}}}{2m}$$

If we assume that the frictional constant f is small, and less than 1, and we set $k = 1$ and $m = 1$, then we realize that the radicand is negative, meaning that our solutions can be expressed more simply as

$$\lambda = -\frac{f}{m} \pm ai$$

Where a is some agglomeration of constants. Since f is positive and m is positive, our eigenvalues will have strictly negative real portions, and thus we can say that our steady state is indeed stable. This is to be expected, as if we were to slightly perturb a spring from its steady state, we wouldn't expect the block to fly into the ground or through the ceiling, but to slowly (and with some oscillation) go back to its steady state.

So here we showed how we would make a higher-order ODE solvable by Matlab, how we'd find a steady state, and how we'd find the stability of the steady state by evaluating the Jacobian of the system at the steady state and finding the eigenvalues of the resulting matrix.