# Two-way ANOVA, I

9.07

4/27/2004

## No class Thursday

- Based upon how we are coming along on the material.
- You shouldn't need the next class to complete the final homework (just posted on the web)
  - Don't need to know how to do post-hoc testing on two-way ANOVA
  - Just read the handout on what post-hoc tests you're allowed to do (confounded vs. unconfounded comparisons)
- Turn in your Thursday HW to one of your TAs

## Two-way ANOVA

- So far, we've been talking about a one-way ANOVA, with one factor (independent variable)
- But, one can have 2 or more factors
- Example: Study aids for exam – how do they affect exam performance?
  - Factor 1: workbook or not
  - Factor 2: 0, 1, or 2 cups of coffee

## Factorial design

| | | Workbook (Factor A) | |
|---|---|---|---|
| | | No | Yes |
| Cups of coffee (Factor B) | 0 | Neither (Control) | Workbook only |
| | 1 | 1 cup coffee only | Workbook + 1 cup coffee |
| | 2 | 2 cups coffee only | Workbook + 2 cups coffee |

Note different levels of Factor A in columns, as when we did one-way ANOVA. Different levels of Factor B in rows.

Each square in the table, representing a particular combination of a level of Factor A and a level of Factor B, is called a *cell*.

## Why do a two-factor (or multi-factor) design?

- Such a design tells us everything about an individual factor that we would learn in an experiment in which it were the only factor
  - The effect of an individual factor, taken alone, on the dependent variable is called a *main effect*
- The design also allows us to study something that we would miss in a one-factor experiment: the *interaction* between the two factors
  - We talked a bit about interactions when we talked about experimental design

## Interactions

- An interaction is present when the effects of one independent variable on the response are different at different levels of the second independent variable.

## Interactions (from an earlier lecture)

- E.G. Look at the effects of aspirin and beta carotene on preventing heart attacks
  - Factors (i.e. independent variables): 1. aspirin, 2. beta carotene
  - Levels of these factors that are tested: 1. (aspirin, placebo), 2. (beta carotene, placebo)

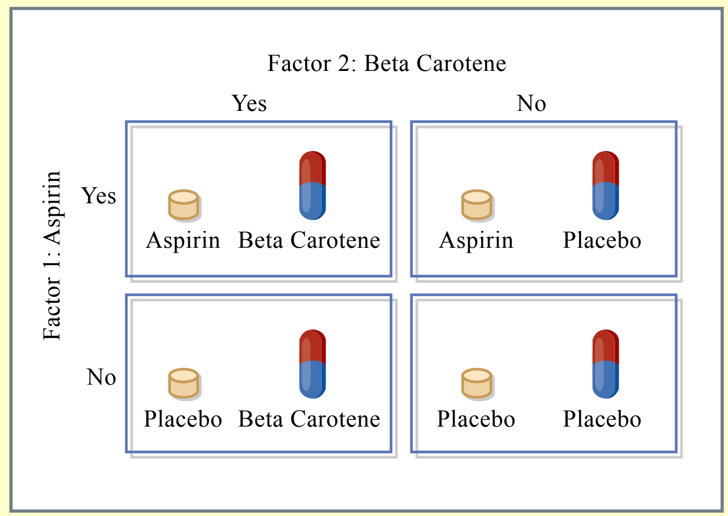


Figure by MIT OCW.

# Outcomes of a factorial design

- Main effects
  - What effect does aspirin have on heart attacks, independent of the level of beta carotene?
  - What effect does beta carotene have on heart attacks, independent of the level of aspirin?
- Interaction(s)
  - The influence that two or more independent variables have on the dependent variable, beyond their main effects
  - How does beta carotene *interact* with aspirin, as far as preventing heart attacks?

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- Does the effect of aspirin on heart attack rates depend upon the level of the beta carotene factor?

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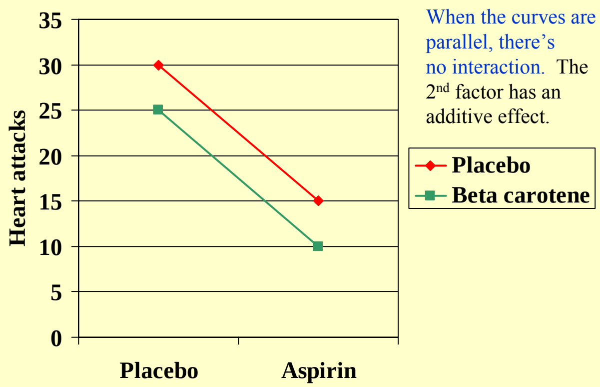
# No interaction




When the curves are parallel, there's no interaction. The 2nd factor has an additive effect.

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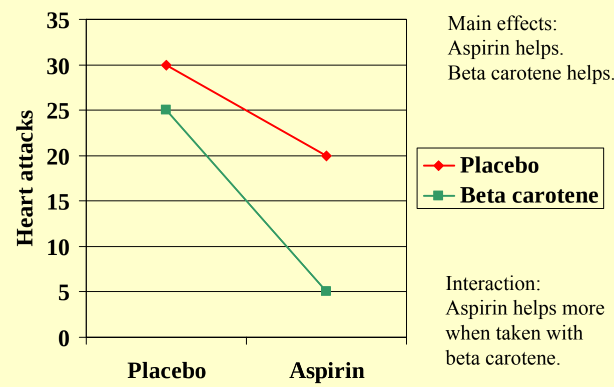
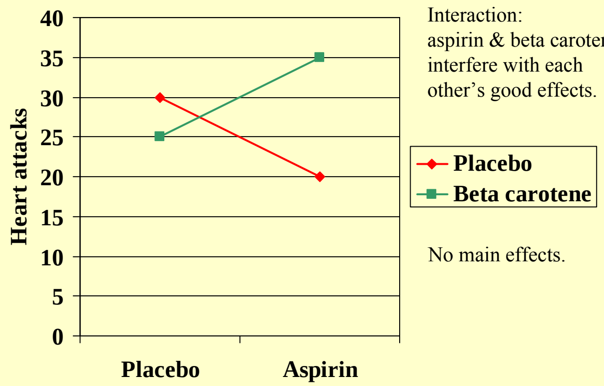
# Interactions




Main effects:
Aspirin helps.
Beta carotene helps.

Interaction:
Aspirin helps more when taken with beta carotene.

## Interactions




Interaction:
aspirin & beta carotene interfere with each other's good effects.

No main effects.

## Why do a two-factor (or multi-factor) design?

- So, for very little extra work, one can study multiple main effects as well as interactions in a single study
  - Multi-factor designs are efficient
- You will often encounter multi-factor designs in behavioral research, in part because we often have hypotheses about interactions

## Between- vs. within-subjects

- As before, the factors could be between- or within-subjects factors, depending upon whether each subject contributed to one cell in the table, or a number of cells
- Also as before, we will start of talking about between-subjects experiments
- In my next lecture we will talk about within-subjects experiments, at least for one-way ANOVAs

## The plan

- Essentially, we're going to split the problem into 3 ANOVAs which look a lot like the one-way ANOVA you've already learned:
  - Main effect ANOVA on factor A
  - Main effect ANOVA on factor B
  - Two-way interaction effect A×B

# The plan

- In each case, we will compute $F_{obt}$ by computing an $MS_{bn}$ specific to the given effect, and dividing it by $MS_{wn}$
- $MS_{wn}$ is a measure of the "noise" – the chance variability which cannot be accounted for by any of the factors.
- We will use the same measure of $MS_{wn}$ for all 3 ANOVAs.

# The plan

- First, we will compute the $SS_{bn}$ for the two main effects, $SS_A$ and $SS_B$, and their degrees of freedom, $df_A$ and $df_B$
- Next, we will compute the $SS_{bn}$ for the interaction, $SS_{A\times B}$, and its degrees of freedom, $df_{A\times B}$
- Then, we will compute $SS_{wn}$, and finally the $F_{obt}$ values $F_A$, $F_B$, and $F_{A\times B}$
- Compare these values with their corresponding critical values, to determine significance

# As in 1-way ANOVA, we'll be filling out a summary table

| Source | Sum of squares | df | Mean square | $F_{obt}$ | $F_{crit}$ |
|---|---|---|---|---|---|
| Between | | | | | |
| Factor A | $SS_A$ | $df_A$ | $MS_A$ | $F_A$ | $F_{crit,A}$ |
| Factor B | $SS_B$ | $df_B$ | $MS_B$ | $F_B$ | $F_{crit,B}$ |
| Interaction | $SS_{A\times B}$ | $df_{A\times B}$ | $MS_{A\times B}$ | $F_{A\times B}$ | $F_{crit, A\times B}$ |
| Within | $SS_{wn}$ | $df_{wn}$ | $MS_{wn}$ | | |
| Total | $SS_{tot}$ | $df_{tot}$ | | | |

# The model

- Score (dependent variable) =
  - Grand mean +
  - Column effect (factor A) +
  - Row effect (factor B) +
  - Interaction effect (A×B) +
  - Error (noise)

## Demonstrating the computations with an example

- Data for our coffee/workbook example (assuming a mere 3 subjects in each condition)

| | | Workbook (Factor A) No | Workbook (Factor A) Yes |
|---|---|---|---|
| Cups of coffee (Factor B) | 0 | 10, 30, 20 | 20, 45, 55 |
| | 1 | 45, 50, 85 | 40, 60, 65 |
| | 2 | 30, 40, 20 | 90, 85, 75 |

## Initial calculations

- As usual, with ANOVA, we're going to want to know $\Sigma x$ and $\Sigma x^2$ for each cell, so we start off calculating those numbers

| | | Workbook (Factor A) No | Workbook (Factor A) Yes |
|---|---|---|---|
| Cups of coffee (Factor B) | 0 | 10, 30, 20<br>$\Sigma x = 60$, $m = 20$<br>$\Sigma x^2 = 1400$ | 20, 45, 55<br>$\Sigma x = 120$, $m = 40$<br>$\Sigma x^2 = 5450$ |
| | 1 | 45, 50, 85<br>$\Sigma x = 180$, $m = 60$<br>$\Sigma x^2 = 11750$ | 40, 60, 65<br>$\Sigma x = 165$, $m = 55$<br>$\Sigma x^2 = 9425$ |
| | 2 | 30, 40, 20<br>$\Sigma x = 90$, $m = 30$<br>$\Sigma x^2 = 2900$ | 90, 85, 75<br>$\Sigma x = 250$, $m = 83.33$<br>$\Sigma x^2 = 20950$ |

## Factor A main effect

- Basically, to analyze the main effect of Factor A (the workbook), analyze the data as if you can just ignore the different levels of Factor B (the coffee)
- Analyze the columns, pretend the rows aren't there

| Workbook (Factor A) | | |
|---|---|---|
| No | Yes | |
| 10, 30, 20<br>$\Sigma x = 60$, m = 20<br>$\Sigma x^2 = 1400$ | 20, 45, 55<br>$\Sigma x = 120$, m = 40<br>$\Sigma x^2 = 5450$ | |
| 45, 50, 85<br>$\Sigma x = 180$, m = 60<br>$\Sigma x^2 = 11750$ | 40, 60, 65<br>$\Sigma x = 165$, m = 55<br>$\Sigma x^2 = 9425$ | |
| 30, 40, 20<br>$\Sigma x = 90$, m = 30<br>$\Sigma x^2 = 2900$ | 90, 85, 75<br>$\Sigma x = 250$, m =83.33<br>$\Sigma x^2 = 20950$ | |
| $\Sigma x = 330$<br>$\Sigma x^2 = 16050$<br>$n_{A1} = 9$ | $\Sigma x = 535$<br>$\Sigma x^2 = 35825$<br>$n_{A2} = 9$ | $\Sigma x = 865$<br>$\Sigma x^2 = 51875$<br>N = 18 |

# Compute $SS_A$

- Here I'll use the computational formula in your handout – it's equivalent to the formula we used for $SS_{bn}$ when talking about one-way ANOVA:

$$SS_A = \Sigma\left(\frac{(\text{Sum of scores in the column})^2}{n \text{ of scores in the column}}\right) - \left(\frac{(\Sigma X_{tot})^2}{N}\right)$$

- $SS_A = 330^2/9 + 535^2/9 - 865^2/18 = 2334.72$

# Compute $df_A$

- This is just like $df_{bn}$ in the one-way ANOVA:

  $df_A$ = (# levels of factor A) – 1 = k – 1 = 1

# Factor B main effect

- Similarly, we analyze the Factor B main effect by essentially ignoring the columns – the different levels of Factor A
- Then, the calculations again look much like they did for a one-way ANOVA

| | | | | |
|---|---|---|---|---|
| Cups of coffee (Factor B) | 0 | 10, 30, 20<br>$\Sigma x = 60$, $m = 20$<br>$\Sigma x^2 = 1400$ | 20, 45, 55<br>$\Sigma x = 120$, $m = 40$<br>$\Sigma x^2 = 5450$ | $\Sigma x = 180$<br>$n_{B1} = 6$ |
| | 1 | 45, 50, 85<br>$\Sigma x = 180$, $m = 60$<br>$\Sigma x^2 = 11750$ | 40, 60, 65<br>$\Sigma x = 165$, $m = 55$<br>$\Sigma x^2 = 9425$ | $\Sigma x = 345$<br>$n_{B2} = 6$ |
| | 2 | 30, 40, 20<br>$\Sigma x = 90$, $m = 30$<br>$\Sigma x^2 = 2900$ | 90, 85, 75<br>$\Sigma x = 250$, $m = 83.33$<br>$\Sigma x^2 = 20950$ | $\Sigma x = 340$<br>$n_{B3} = 6$ |
| | | | | $\Sigma x = 865$<br>$\Sigma x^2 = 51875$<br>$N = 18$ |

## Compute $SS_B$

- This is the exact same formula as the one for $SS_A$, just applied to the other factor:

$$SS_B = \Sigma\left(\frac{(\text{Sum of scores in the row})^2}{n \text{ of scores in the row}}\right) - \left(\frac{(\Sigma X_{tot})^2}{N}\right)$$

- $SS_B = 180^2/6 + 345^2/6 + 340^2/6 - 865^2/18 =$ 2936.11

## Compute $df_B$

- Again, this is just like $df_{bn}$ in the one-way ANOVA:

  $df_B$ = (# levels of factor B) – 1 = k – 1 = 2

## OK, now a trickier one: the interaction

- Differences between cells are a result of the main effects for factors A and B, and the interaction between A and B
- The overall sum of squares between cells ($SS_{bn}$) equals $SS_A + SS_B + SS_{A\times B}$
- So, $SS_{A\times B} = SS_{bn} - SS_A - SS_B$

## Computing $SS_{bn}$

- This is basically treating the cells like they each come from a different level of a single factor, then doing the same computation as for $SS_A$ and $SS_B$

$$SS_{bn} = \Sigma\left(\frac{(\text{Sum of scores in the cell})^2}{n \text{ of scores in the cell}}\right) - \left(\frac{(\Sigma X_{tot})^2}{N}\right)$$

- $SS_{bn} = 60^2/3 + 120^2/3 + 180^2/3 + 165^2/3 + 90^2/3 + 250^2/3 - 865^2/18 = 7840.28$

| | |
|---|---|
| 10, 30, 20<br>$\Sigma x = 60$ | 20, 45, 55<br>$\Sigma x = 120$ |
| 45, 50, 85<br>$\Sigma x = 180$ | 40, 60, 65<br>$\Sigma x = 165$ |
| 30, 40, 20<br>$\Sigma x = 90$ | 90, 85, 75<br>$\Sigma x = 250$ |

## Computing $SS_{A\times B}$

- $SS_{A\times B} = SS_{bn} - SS_A - SS_B$
  $= 7840.28 - 2334.72 - 2936.11$
  $= 2569.45$

## Compute $df_{A\times B}$

- Similar logic for $SS_{bn}$ gives us
  $df_{bn} = df_A + df_B + df_{A\times B}$
  $df_{A\times B} = df_{bn} - df_A + df_B$
- $df_{bn} = k_{bn} - 1 = \#\text{ cells} - 1 = k_A k_B - 1$
- $df_{A\times B} = (k_A k_B - \cancel{1}) - (k_A - \cancel{1}) - (k_B - 1)$
  $= k_A(k_B - 1) - (k_B - 1)$
  $= (k_A - 1)(k_B - 1) = df_A \cdot df_B$
  $= 1 \cdot 2 = 2$

## Let's see what we've got so far

| Source | Sum of squares | df | Mean square | F | $F_{crit}$ |
|---|---|---|---|---|---|
| Between | | | | | |
| Factor A | 2334.72 | 1 | 2334.72 | $F_A$ | $F_{crit,A}$ |
| Factor B | 2936.11 | 2 | 1468.06 | $F_B$ | $F_{crit,B}$ |
| Interaction | 2569.45 | 2 | 1284.73 | $F_{A\times B}$ | $F_{crit,\ A\times B}$ |
| Within | $SS_{wn}$ | $df_{wn}$ | $MS_{wn}$ | | |
| Total | $SS_{tot}$ | $df_{tot}$ | | | |

## $MS_{wn}$

- What we really need is $MS_{wn}$, the measure of the "noise", the chance variation unexplained by either of the effects or their interaction
- This can be computed directly, but as your handout suggests, it's probably easier to use:

  $SS_{wn} = SS_{tot} - SS_{bn}$

  $df_{wn} = N - k_{A\times B} = N - \text{(number of cells)}$

## Computing $SS_{tot}$

- As with one-way ANOVA,

$$SS_{tot} = (\sum x^2)_{tot} - \frac{(\sum x)^2_{tot}}{N_{tot}}, \quad \text{df} = \text{N - 1}$$

- We had already computed $\Sigma x^2$ for each cell, and added them up.
- $SS_{tot} = 51875 - 865^2/18 = 10306.94$

## Computing $SS_{wn}$

- $SS_{wn} = SS_{tot} - SS_{bn} = 10306.94 - 7840.28$
  $= 2466.66$

## Back to the summary table

| Source | Sum of squares | df | Mean square | F | $F_{crit}$ |
|---|---|---|---|---|---|
| Between | | | | | |
| Factor A | 2334.72 | 1 | 2334.72 | $F_A$ | $F_{crit,A}$ |
| Factor B | 2936.11 | 2 | 1468.06 | $F_B$ | $F_{crit,B}$ |
| Interaction | 2569.45 | 2 | 1284.73 | $F_{A\times B}$ | $F_{crit, A\times B}$ |
| Within | 2466.66 | 12 | $MS_{wn}$ | | |
| Total | 10306.94 | 17 | | | |

## Back to the summary table

| Source | Sum of squares | df | Mean square | F | $F_{crit}$ |
|---|---|---|---|---|---|
| Between | | | | | |
| Factor A | 2334.72 | 1 | 2334.72 | 11.36 | $F_{crit,A}$ |
| Factor B | 2936.11 | 2 | 1468.06 | 7.14 | $F_{crit,B}$ |
| Interaction | 2569.45 | 2 | 1284.73 | 6.25 | $F_{crit, A\times B}$ |
| Within | 2466.66 | 12 | 205.56 | | |
| Total | 10306.94 | 17 | | | |

## Getting the $F_{crit}$ values

- This is much like in one-way ANOVA
- Look up $F_{crit}$ in an F-table, with df from the numerator and denominator of $F_{obt}$
- $F_{crit}$ for $F_A$ has ($df_A$, $df_{wn}$) degrees of freedom
- $F_{crit}$ for $F_B$ has ($df_B$, $df_{wn}$) degrees of freedom
- $F_{crit}$ for $F_{A\times B}$ has ($df_{A\times B}$, $df_{wn}$) degrees of freedom

- Here, we will use $\alpha$=0.05

## $F_{obt}$'s & $F_{crit}$'s

- Main effect of workbook:
  - $F_A = 11.36$
  - $F_{0.05,1,12} = 4.75$ Significant
- Main effect of coffee:
  - $F_B = 7.14$
  - $F_{0.05,2,12} = 3.88$ Significant
- Interaction:
  - $F_{A\times B} = 6.25$
  - $F_{crit,2,12} = 3.88$ Significant

## Results

- Both main effects and their interaction are significant
  - Use of the workbook to study for the exam had a significant effect on exam performance ($F(1,12) = 11.36$, $p<0.05$).
  - Drinking coffee also had a significant effect on exam performance ($F(2,12) = 7.14$, $p<0.05$)
  - And the interaction between coffee drinking and workbook use was significant ($F(2,12) = 6.25$, $p<0.05$)

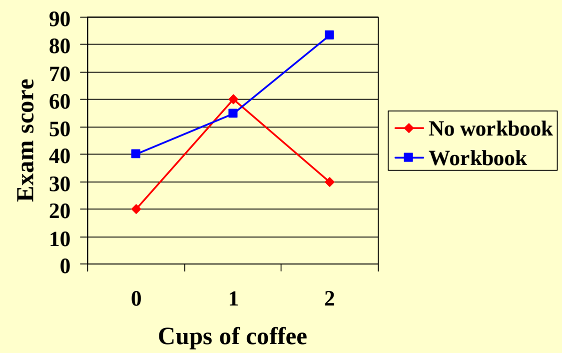
## Graphing the results

- Main effects are often simple enough that you can understand them without a graph (though you certainly can graph them)
- Means for factor A:
  - No workbook: 36.67, Workbook: 59.44
- Means for factor B:
  - 0 coffee: 30, 1 cup: 57.5, 2 cups: 56.67

## Graphing the results: interaction

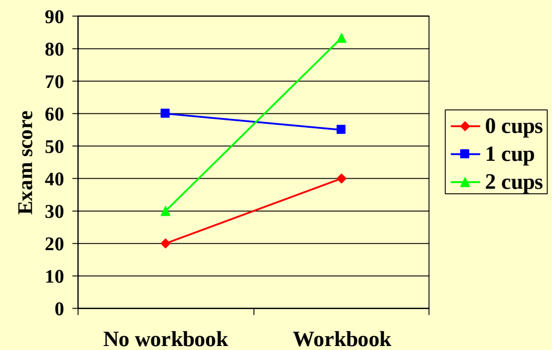
- Interactions are tricky – graph them to see what's going on!
- For each cell, plot the mean
- Plot the factor with more levels on the x-axis, dependent variable on the y-axis
- Connect points corresponding to the same level of the other factor

| | | Workbook (Factor A) | |
|---|---|---|---|
| | | No | Yes |
| Cups of coffee (Factor B) | 0 | m = 20 | m = 40 |
| | 1 | m = 60 | m = 55 |
| | 2 | m = 30 | m = 83.3 |



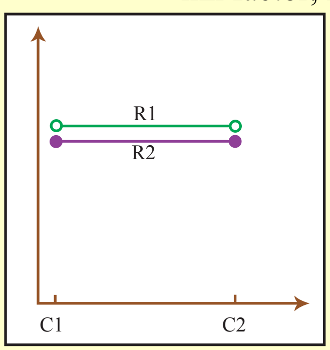
## Why plot the factor with more levels on the x-axis?

- This is good plotting style
- We humans are not so good at understanding plots with lots of lines in them, unless those lines are parallel or have some other simple relationship to each other
  - The difference between 2 & 3 lines is trivial, but this becomes more important if one factor has ≥ 4 levels
- Nonetheless, it can sometimes be instructive to plot it the other way:




## Outcomes of factorial ANOVAs: (Nearly) parallel lines indicate an insignificant interaction

mn factor, R = row factor

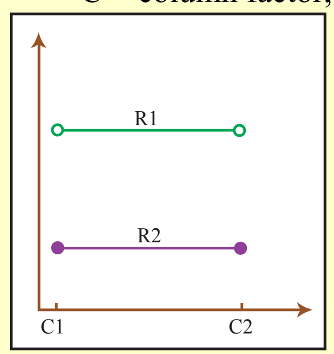



C not significant
R not significant

Figure by MIT OCW.

## Outcomes of factorial ANOVAs: (Nearly) parallel lines indicate an insignificant interaction

- C = column factor, R = row factor

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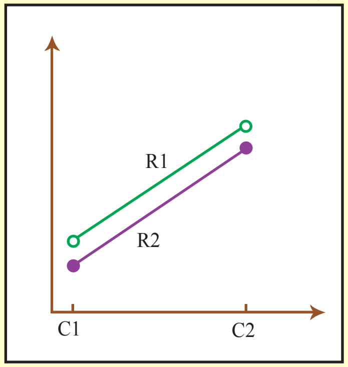


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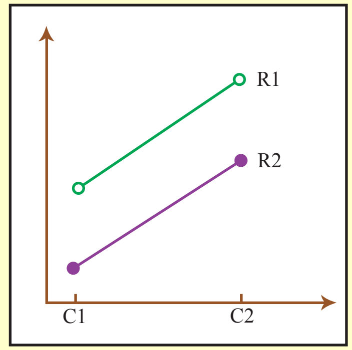
## Outcomes of factorial ANOVAs: (Nearly) parallel lines indicate an insignificant interaction

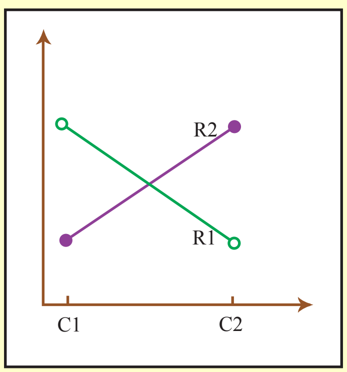
- C = column factor, R = row factor




C significant
R significant

Figure by MIT OCW.

## Outcomes of factorial ANOVAs: non-parallel lines indicate significant interaction

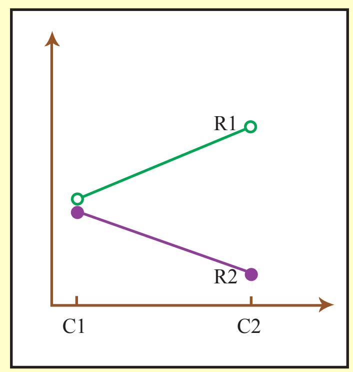



C not significant
R not significant

*A main effect is significant if the mean for one level of the factor is sufficiently different from the mean for another level of the factor*

Figure by MIT OCW.

## Outcomes of factorial ANOVAs: non-parallel lines indicate significant interaction

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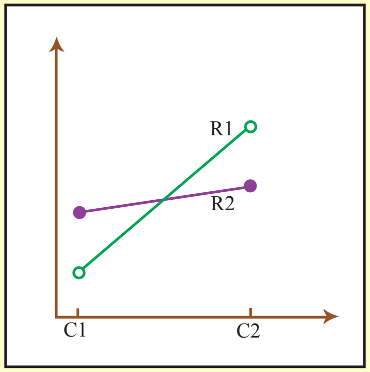


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R not significant

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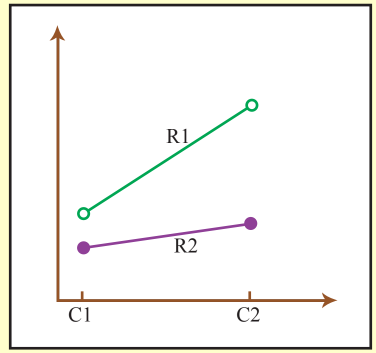
## Outcomes of factorial ANOVAs: non-parallel lines indicate significant interaction



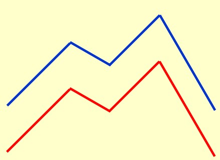

C significant
R significant

*A main effect is significant if the mean for one level of the factor is sufficiently different from the mean for another level of the factor*

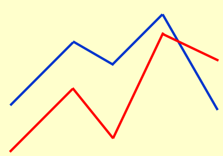
Figure by MIT OCW.

## With more than two levels

- No interaction



- Significant interaction



When the lines *cross*, that's a sign you have an interaction (it may not be significant, however, so you need to check)

## Interpreting the results

- If the interaction is significant, the main effects must be interpreted with care
- E.G. we do not just conclude, "look, the workbook helped", since whether or not it helped depends upon how much coffee the student drank

## Summary

- We've talked about how to perform a two-way ANOVA
- And we've looked at what the graphs of the data might look like for different combinations of main effects and interactions
- Stepping back for a moment…

## Assumptions of the two-way ANOVA

- Between-subjects: the sample in each cell (i.e. for each combination of levels of the two factors) is independent of the samples in the other cells
- The sample in each cell comes from an (approximately) normal distribution
- The populations corresponding to each cell have the same variance (homogeneous variance assumption)

## Complete vs. incomplete ANOVA

- Furthermore, we were assuming that the ANOVA was *complete*, meaning that all levels of factor A were combined with all levels of factor B
  - Incomplete factorial designs require more elaborate procedures than the one we've just used

## What were the null hypotheses?

- Main effects:

  $H_0$: $\mu_{A1} = \mu_{A2}$

  $H_0$: $\mu_{B1} = \mu_{B2} = \mu_{B3}$

  $H_a$: means not all equal
- Interaction:

  $H_0$: There is no interaction effect in the population – regardless of the level of, say, factor B, a change in factor A leads to the same difference in mean response

  $\mu_{A1B1}-\mu_{A2B1} = \mu_{A1B2}-\mu_{A2B2} = \mu_{A1B3}-\mu_{A2B3}$

  $H_a$: Not all these differences are equal

## Homework comments

- Where it says "describe what the graph would look like," just plot the graph
- Where it refers to "estimating the effect sizes", what they mean is:
  - Main effect: mean(level i) – (grand mean)
  - Interaction: mean(cell ij) – (grand mean)
- Problem labeled "9" (not the 9th problem): based on the results of the previous problem, how many post-hoc tests will you want to do? (Read the handout on confounded vs. unconfounded tests). Use this to estimate the experiment-wise error rate based on the per-comparison rate.