Probability I

9.07
2/10/2004

Class details

- Reminder: HW 1 due on Friday.
- HW2 is now on the web. It's due Friday of next week.
- Readings in Probability now on the web.
- Reminder: Office hours today, 3-4 pm

Probability and gambling

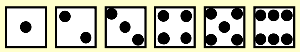
- De Mere: "Which is more likely, rolling at least one 6 in 4 rolls of a single die, or rolling at least one double 6 in 24 rolls of a pair of dice?"
- De Mere reasoned they should be the same:
  - Chance of one 6 in one roll = 1/6
  - Average number in 4 rolls = 4·(1/6) = 2/3
  - Chance of one double 6 in one roll = 1/36
  - Average number in 24 rolls = 24·(1/36) = 2/3
- Why, then, did it seem like he lost more often with the second gamble?
- He asked his friend Pascal, and Pascal & Fermat worked out the theory of probability.

Basic definitions

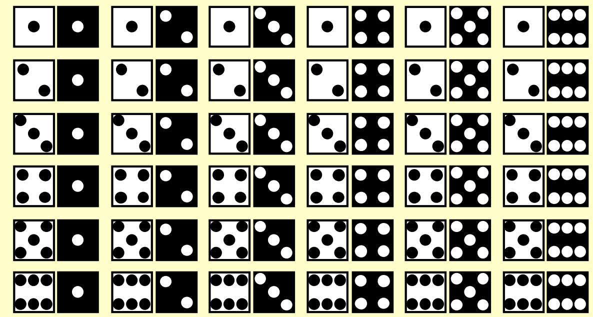
- Random experiment = observing the outcome of a chance event.
- Elementary outcome = any possible result of the random experiment = O_i
- Sample space = the set of all elementary outcomes.

Example sample spaces

- Tossing a single coin:
  - {H, T}
- Tossing two coins:
  - {HH, TH, HT, TT}
- One roll of a single die:



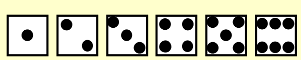
Sample space for a pair of dice



Each pair is an elementary outcome.

Fair coin or die

- For a fair coin or die, the elementary outcomes have equal probability
  - P(H) = P(T) = 0.5
  - P(1 spot) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6
- Of course, the coin or die might not be fair



P = .15 .10 .25 .15 .15 .20

Properties of probabilities

- $P(O_i) \geq 0$
  - Negative probabilities are meaningless
- The total probability of the sample space must equal 1.
  - If you roll the die, one of the elementary outcomes must occur.

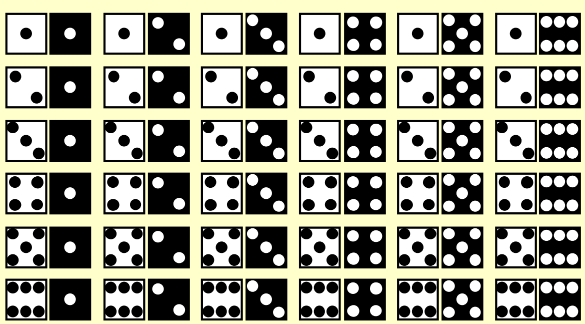
How do we decide what these probabilities are?

- 1. Probability = event's relative frequency in the *population.*
  - Look at every member of the population, and record the relative frequency of each event.
  - Often you simply can't do this.
- 2. Estimate probability based on the relative frequency in a (large) sample.
  - Not perfect, but feasible.
- 3. Classical probability theory: probability based on an assumption that the game is fair.
  - E.G. heads and tails equally likely.
  - Similarly, might otherwise have a theoretical model for the probabilities.

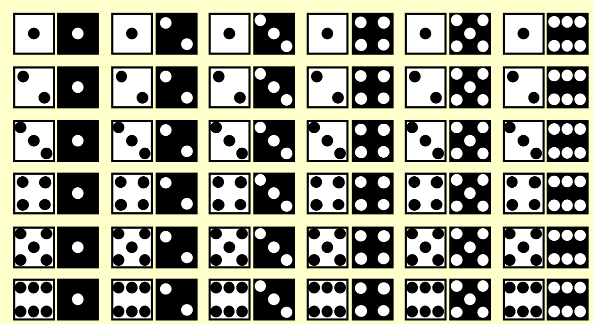
Events

- An event is a set of elementary outcomes.
- The probability of an event is the sum of the probabilities of the elementary outcomes.
- E.G. tossing a pair of dice:

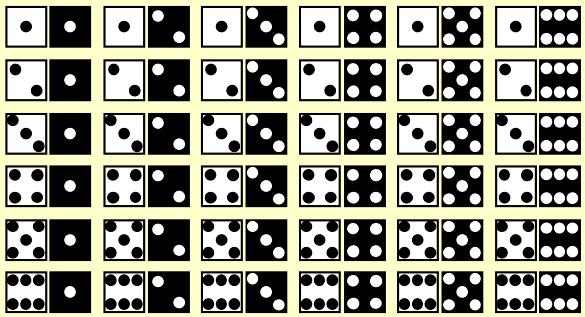
Event A: Dice sum to 3



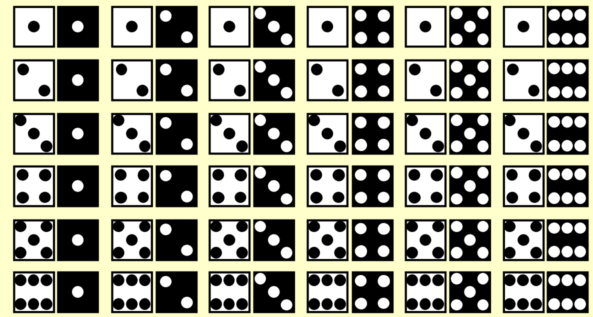
Event B: Dice sum to 6



Event C: White die = 1



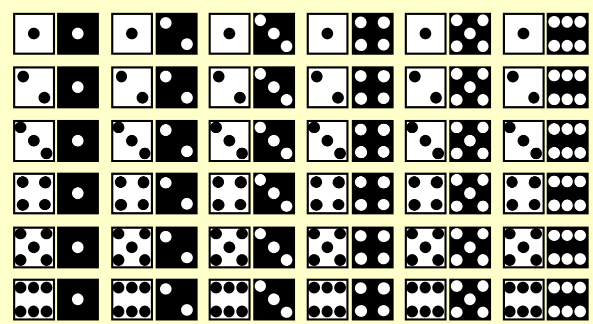
Event D: Black die = 1



Combining events

- E AND F: both event E and event F occur
- E OR F: either event E occurs, or event F does, or both
- NOT E: event E does not occur

C OR D: W=1 OR B=1



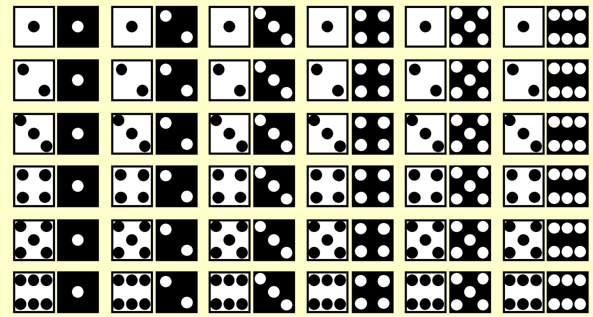
The addition rule

- P(W=1) = 6/36
- P(B=1) = 6/36
- P(W=1 or B=1) ≠ P(W=1) + P(B=1)

- P(E or F) = P(E) + P(F) – P(E and F)

Subtract the region of overlap, so you don't count it twice.

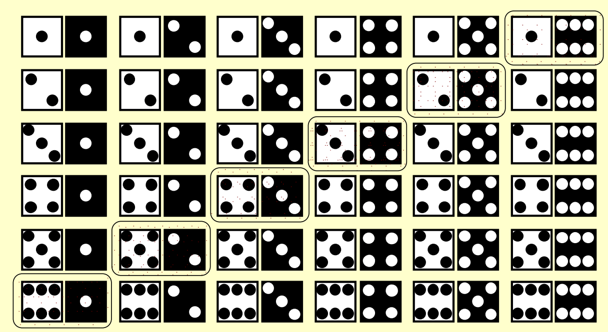
A or B: Dice sum to 3, or sum to 6



Mutually exclusive events

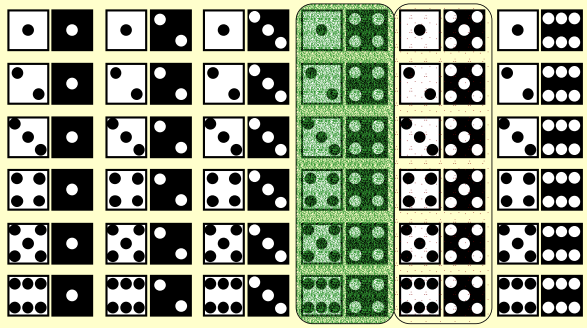
- Events E and F are *mutually exclusive* if the two events could *not* have both occurred.
  - P(E and F) = 0.
  - The events have no elementary outcomes in common. (There's no overlap in our sample space diagram.)
- If E and F are mutually exclusive,
  - P(E or F) = P(E) + P(F)
- The elementary outcomes are mutually exclusive.
  - P(any O_i) = $P(O_1)+P(O_2)+\cdots P(O_N) = 1$

Another example: P(sum=7) = ?



P(sum=7) = 6/36.

Another example: P(B=5 or 4) = ?

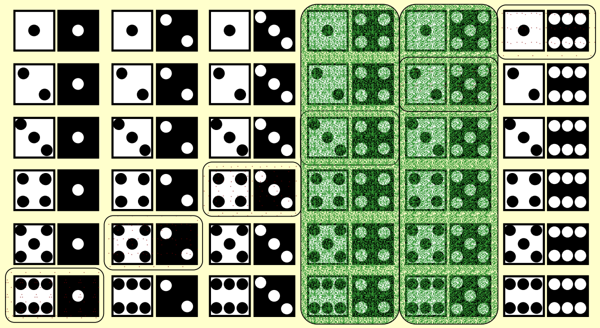


P(B=5 or 4) = 6/36 + 6/36 = 12/36.

P(sum=7 or (B=5 or 4)) = ?

- P(sum=7 and (B=5 or 4)) =
  P({2, 5}, {3, 4}) = 2/36
- P(sum=7 or (B=5 or 4)) =
  6/36 + 12/36 – 2/36 = 16/36

P(sum=7 or (B=5 or 4))



P(A or B) = P(A) + P(B) – P(A and B) = 6/36 + 12/36 – 2/36 = 16/36.

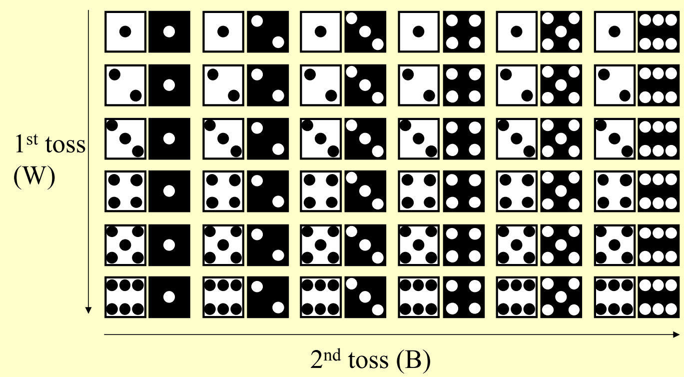
De Mere revisited

- Wanted to know what is the probability of getting at least one 6 in 4 tosses of a die.
- P(1^{st}=6 or 2^{nd}=6 or 3^{rd}=6 or 4^{th}=6)
- P(1^{st}=6) = P(2^{nd}=6) = P(3^{rd}=6)=P(4^{th}=6)
  = 1/6
- Are these events mutually exclusive?
  - No, you could get a 6 on both the 1^{st} & 2^{nd} tosses, for example.
  - So De Mere was incorrect. $P \neq 4 \cdot (1/6)$

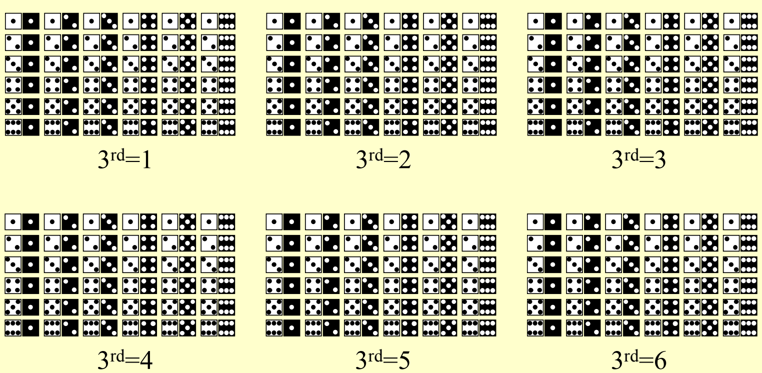
The addition formula, continued

- P(E or F) = P(E) + P(F) – P(E and F)
- You will probably rarely use this formula except for simple cases! It gets complicated quickly if you want to compute
  P(E or F or G or …)
- Example:
  P(at least one 6 in 3 tosses of a die)=?

2 tosses of a die

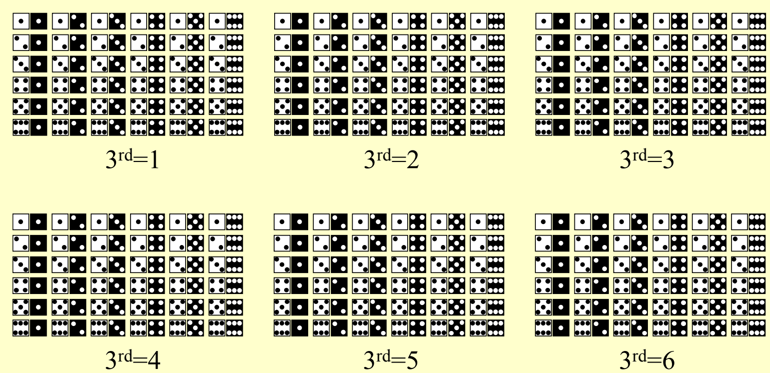


3 tosses of a die



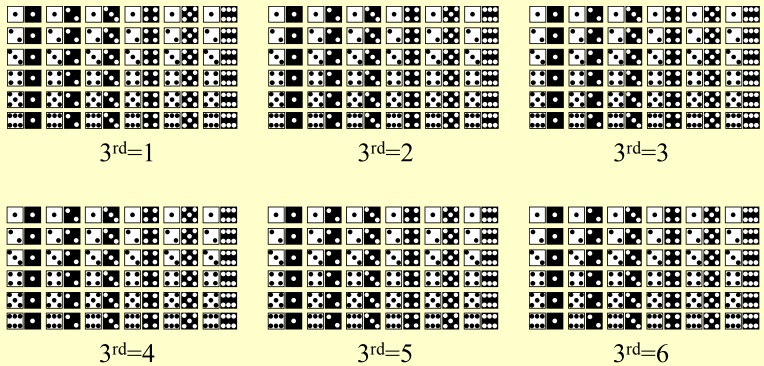
P(one 6 in 3 rolls) = P(1^{st}=6) + P(2^{nd}=6) + P(3^{rd}=6) +…

3 tosses of a die



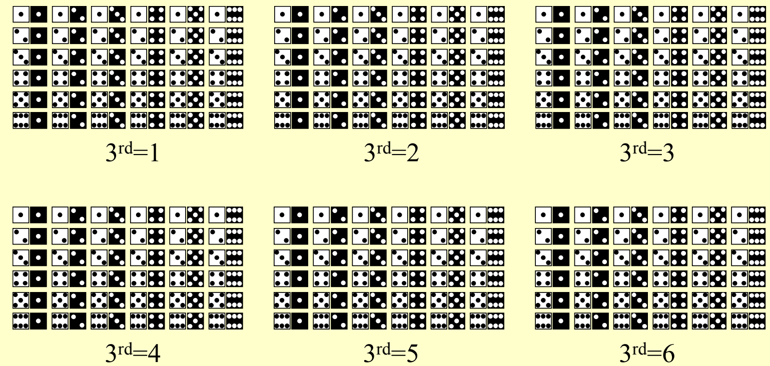
P(one 6 in 3 rolls) = ... – P(6 in 1^{st} & 2^{nd}) – …

3 tosses of a die



P(one 6 in 3 rolls) = ... – P(6 in 1st & 3rd) – …

3 tosses of a die

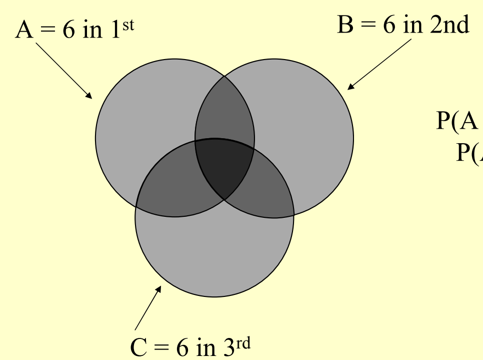


P(one 6 in 3 rolls) = ... – P(6 in 2nd & 3rd) – …

3 tosses of a die

- P(at least one 6 in 3 tosses) =
  P(6 in 1st) + P(6 in 2nd) + P(6 in 3rd) –
  P(6 in 1st & 2nd) – P(6 in 1st & 3rd) –
  P(6 in 2nd & 3rd) + P(6 in 1st & 2nd & 3rd)
  = 1/6+1/6+1/6-1/36-1/36-1/36+1/216
  = 91/216
- Phew... It only gets worse from here. De Mere probably doesn't want to calculate P(at least one 6 in 4 tosses) this way. Luckily there are other ways to go about this.

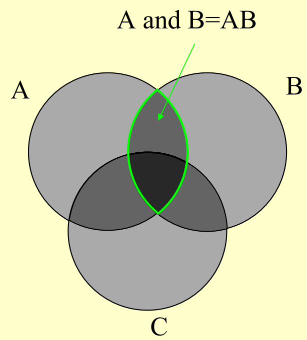
3 tosses of a die – Venn diagram



"or"

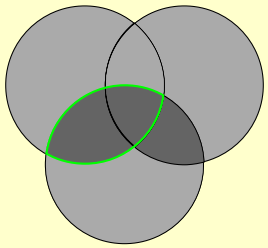
P(A + B + C) =
P(A) + P(B) + P(C) …

3 tosses of a die – Venn diagram



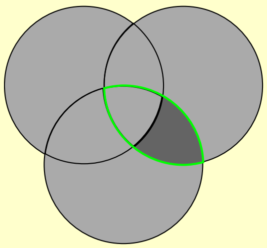
P(A + B + C) =
P(A) + P(B) + P(C)
– P(AB)…

3 tosses of a die – Venn diagram



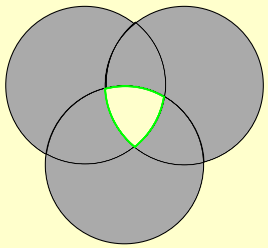
P(A + B + C) =
P(A) + P(B) + P(C)
– P(AB) – P(AC)…

3 tosses of a die – Venn diagram



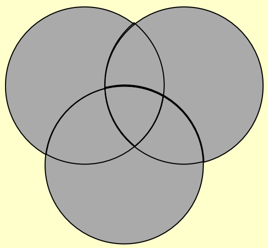
P(A + B + C) =
P(A) + P(B) + P(C)
– P(AB) – P(AC)
– P(BC)…

3 tosses of a die – Venn diagram



P(A + B + C) =
P(A) + P(B) + P(C)
– P(AB) – P(AC)
– P(BC) + P(ABC)

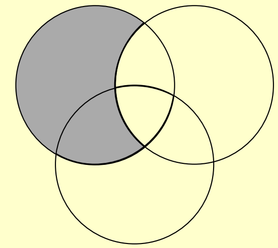
3 tosses of a die – Venn diagram



P(A + B + C) =
P(A) + P(B) + P(C)
– P(AB) – P(AC)
– P(BC) + P(ABC)

Venn diagrams, II

- Just as with sample space diagrams, lack of overlap means two events are mutually exclusive.
- Consider the event "A, but not A and B)" = A-AB.
- Are the events B, and A-AB mutually exclusive?

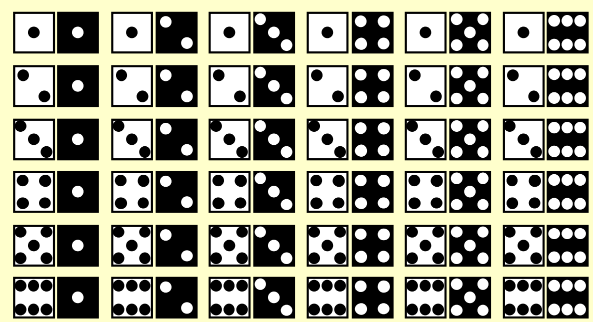


Yes.

Conditional probability

- The probability that event A will occur, given that event C has already occurred.
- P(A|C)
- P(dice sum to 3) = P({1,2},{2,1}) = 2/36.
- Suppose we have already tossed the black die, and got a 2. Given that this has already occurred, what is the probability that the dice will sum to 3?

P(A|C) = P(sum to 3|B=2)



Another formula

- P(A|C) = P(A and C)/P(C)
- E.G.

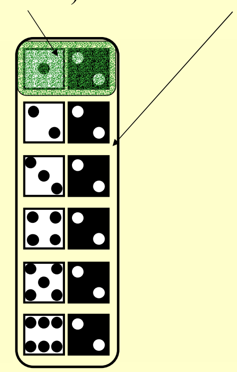
  P(sum to 3|B=2) =

  P(B=2 & sum to 3)/P(B=2)

The formula in action

P(A & C) = 1/36

P(C) = 1/6



P(A|C) = (1/36) / (1/6) = 1/6.

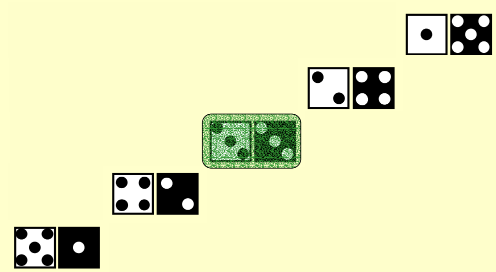
Rearranging to get the multiplication rule

- P(E|F) = P(E and F)/P(F)
- Multiplication rule:

  P(E and F) = P(F) P(E|F)

- Another example: What is the probability that the sum=6 and the white die came up a 3?

P(sum=6 and W=3) = P(sum=6) P(W=3|sum=6)

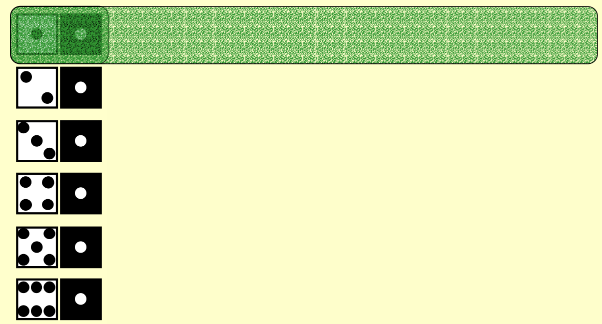


P(sum=6)=5/36. P(W=3|sum=6)=1/5. P(sum=6 & W=3)=1/36.

Some notes

- P(E|E) = P(E and E)/P(E) = P(E)/P(E) = 1
  - Once an event occurs, it's certain.
- If E and F are mutually exclusive,
  P(E|F) = P(E and F)/P(F) = 0/P(F) = 0
  - Once F has occurred, E is impossible, because the two are mutually exclusive.
- Swapping E & F
  -> P(F|E) = P(E and F)/P(E)
  -> P(F|E) P(E) = P(E and F) = P(E|F) P(F)
  -> P(E) P(F|E) = P(F) P(E|F)
- Another example: what is P(W=1 | B=1)?

P(W=1|B=1)



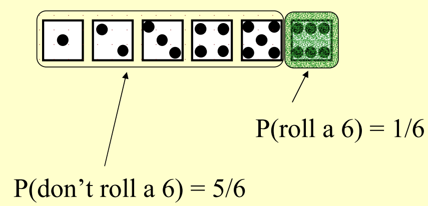
P(W=1|B=1) = 1/6 = P(W=1).

Independent events

- Two events E and F are independent if the occurrence of one has no effect on the probability of the other.
- E.G. the roll of one die has no effect on the roll of another (unless they're glued together or something).
- If E and F are independent, this is equivalent to saying that P(E|F) = P(E), and P(F|E) = P(F)
- Special multiplication rule for independent events: P(E and F) = P(E) P(F)

A last rule (an easy one)

- P(not E) = 1-P(E)
  - e.g. P(failed to roll a 6)



Probability rules

- Addition rule:
  - P(E or F) = P(E) + P(F) – P(E and F)
- Addition rule for E & F mutually exclusive:
  - P(E or F) = P(E) + P(F)
- Multiplication rule:
  - P(E and F) = P(E|F) P(F) = P(F|E) P(E)
- Multiplication rule, independent E & F:
  - P(E and F) = P(E) P(F)
- Inverse rule:
  - P(not E) = 1 – P(E)

Now we're ready to solve De Mere's problem (the easy way)

- What is the probability of getting at least one 6 on 4 rolls of a die?
- Remember how icky the addition rule got for P(A or B or C or D)? Problems like this are often easier to solve in reverse. Find the probability of the event NOT happening.
  - But sometimes figuring out what the inverse is can be tricky

What is the probability of getting at least one 6 one 4 rolls of a die?

- P(not E) = P(roll 4 times and don't roll a 6)
- P(don't roll a 6 on one roll) = 5/6
- We know rolls are independent, so P(don't roll a 6 on any roll) =
  P(no 6 on 1st) · P(no 6 on 2nd) ·
  P(no 6 on 3rd) · P(no 6 on 4th) =
  $(5/6)^4$ = 0.482
- P(E) = 1 – P(not E) = 1-0.482 = 0.518

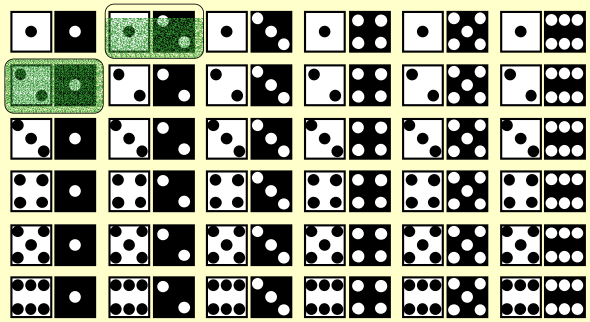
What is the probability of getting at least one pair of 6's on 24 rolls of a pair of dice?

- Again, solve the problem in reverse.
- P(not E) = P(no pair of 6's on any of 24 rolls)
- P(pair of 6's on a single roll) = 1/36.
- P(no pair of 6's on a single roll) = 35/36.
- P(not E) = $(35/36)^{24}$ = 0.509
- P(E) = 1 – P(not E) = 1 – 0.509 = 0.491

De Mere was right – this event is less likely to occur than rolling at least one 6 in 4 throws. (It's a pretty small difference – he must have gambled a lot and paid close attention to the results!)

P(loaded dice sum to 3)=?

P({1,2,3,4,5,6}) = {.15, .10, .25, .15, .15, .20}



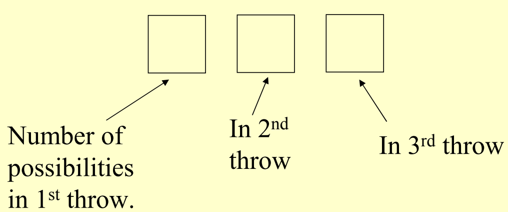
P(sum to 3) = 0.15*0.10 + 0.10*0.15 = 0.03

Another way to solve problems with fair dice & coins

- Probability of event made up of equally probable elementary outcomes = (# of outcomes that are part of the event)/(total number of outcomes)
- E.G. P(at least one 6 in 3 throws)
- How many total outcomes?
- How many outcomes with at least one 6?

A useful visualization

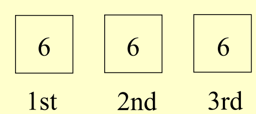
- Box diagrams:



- To get total number of possible outcomes, multiply the numbers in the 3 boxes.

Total number of outcomes in 3 throws of a die

- 6 possibilities for each throw



- Total number of outcomes from 3 throws = 6*6*6 = 216.

Number of outcomes including at least one 6

- First, assume the 1st throw is a 6:

1	6	6
1st	2nd	3rd

Only one possibility – it's a 6

These can be anything they like. 6 possibilities each.

Number of outcomes including at least one 6

- Next, assume the 2nd throw is a 6:

5	1	6
1st	2nd	3rd

5 possibilities – we already counted the outcomes where this is a 6.

This can be anything. 6 possibilities.

One possibility – it's a 6

Number of outcomes including at least one 6

- Finally, assume the 3rd throw is a 6:

5	5	1
1st	2nd	3rd

5 possibilities each – we already counted the outcomes where these are 6's.

One possibility – a 6

Another fairly easy and reliable way to solve problems like this

- Probability of event made up of equally probable elementary outcomes = (# of outcomes that are part of the event)/(total number of outcomes)
- E.G. P(at least one 6 in 3 throws)
- How many total outcomes?
  - 6*6*6 = 216
- How many outcomes with at least one 6?
  - 1*6*6 + 5*1*6 + 5*5*1 = 36+30+25 = 91
- P(E) = 91/216.

Check: do the problem the other way

- P(no 6 in 3 throws) = $(5/6)^3$
- P(at least one 6 in 3 throws) =
  $1 - (5/6)^3 = 216/216 - 125/216 = 91/216$
- It worked!