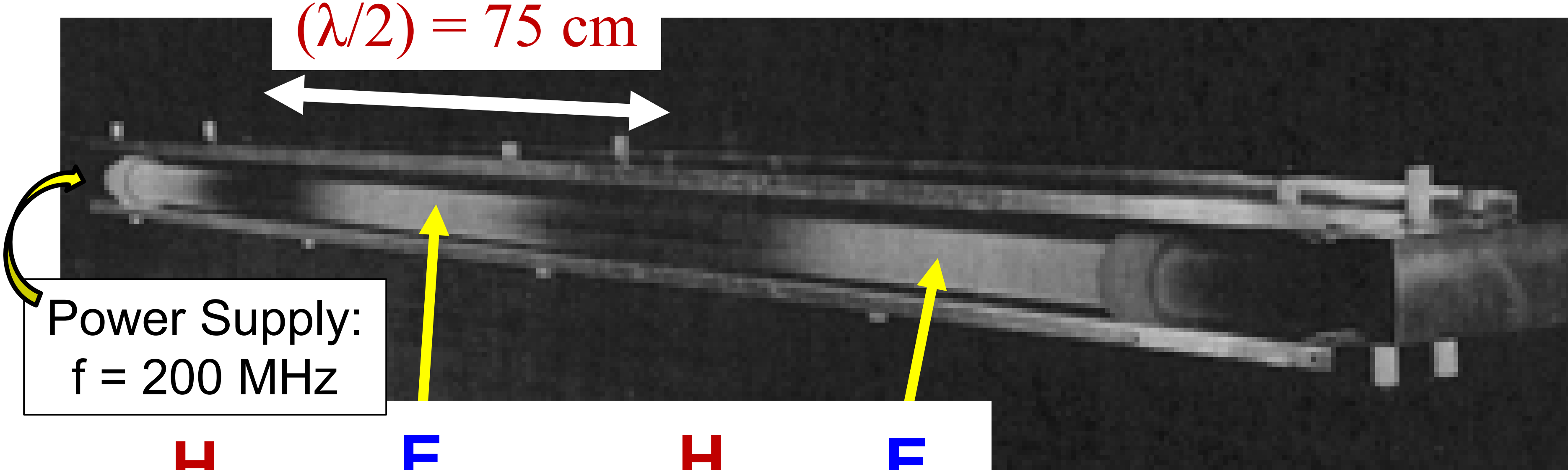
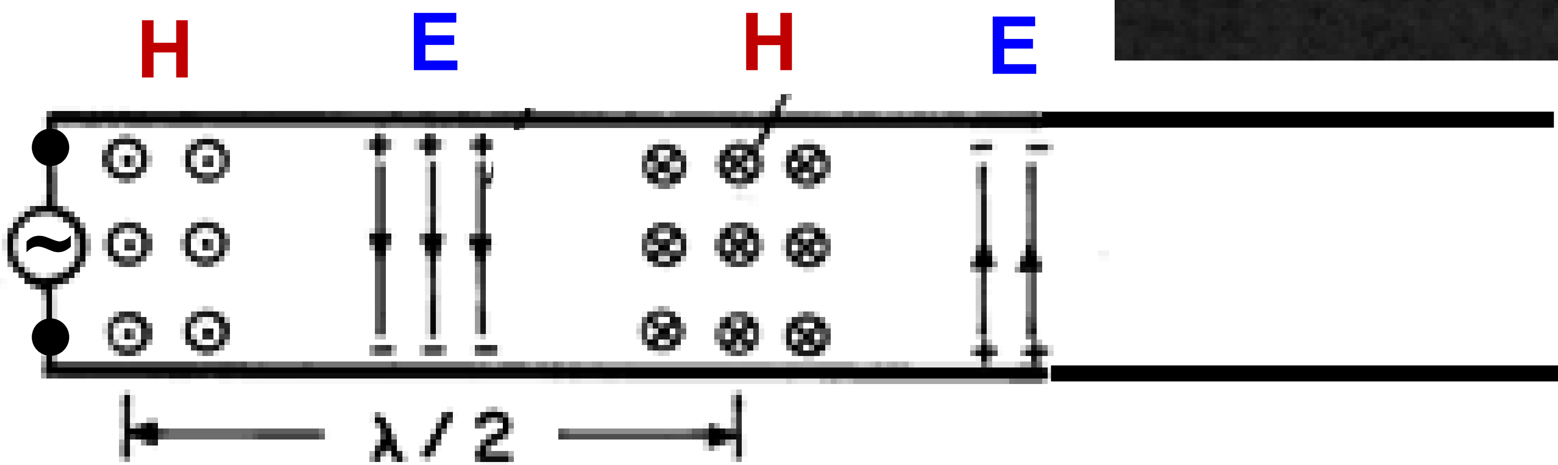
Table 2.7 Maxwell's equations for linear media.

Name	Differential form
Gauss' law	$\nabla \cdot \epsilon \boldsymbol{E} = \rho_e$
Ampère's law	$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \epsilon \boldsymbol{E}}{\partial t}$
Faraday's law	$\nabla \times \boldsymbol{E} = -\frac{\partial \mu \boldsymbol{H}}{\partial t}$
Magnetic flux	$\nabla \cdot \mu \boldsymbol{H} = 0$
Charge conservation	$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho_e}{\partial t}$

EM Waves

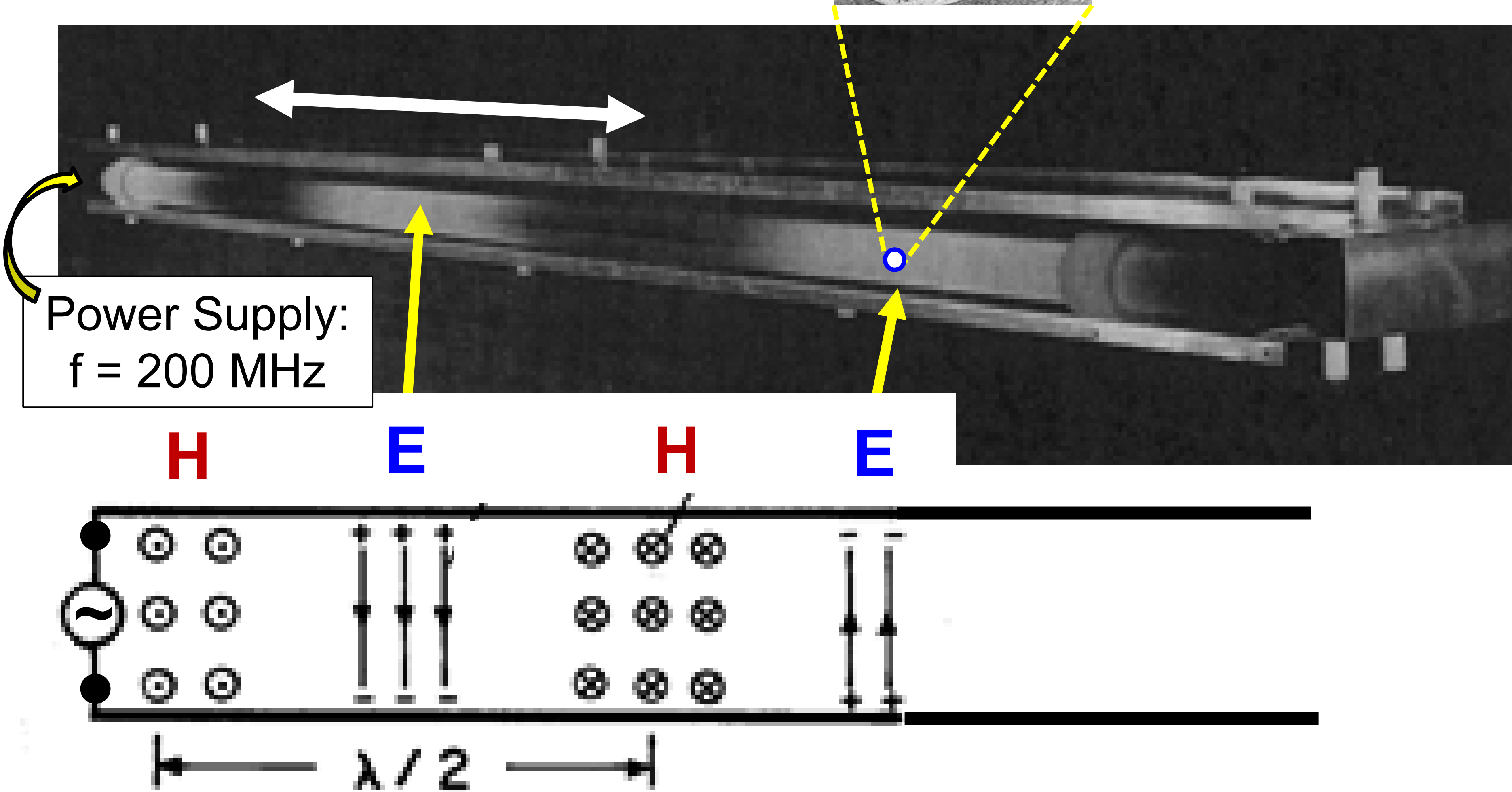
Demo from previous Lecture:
Electromagnetic "Standing Wave"

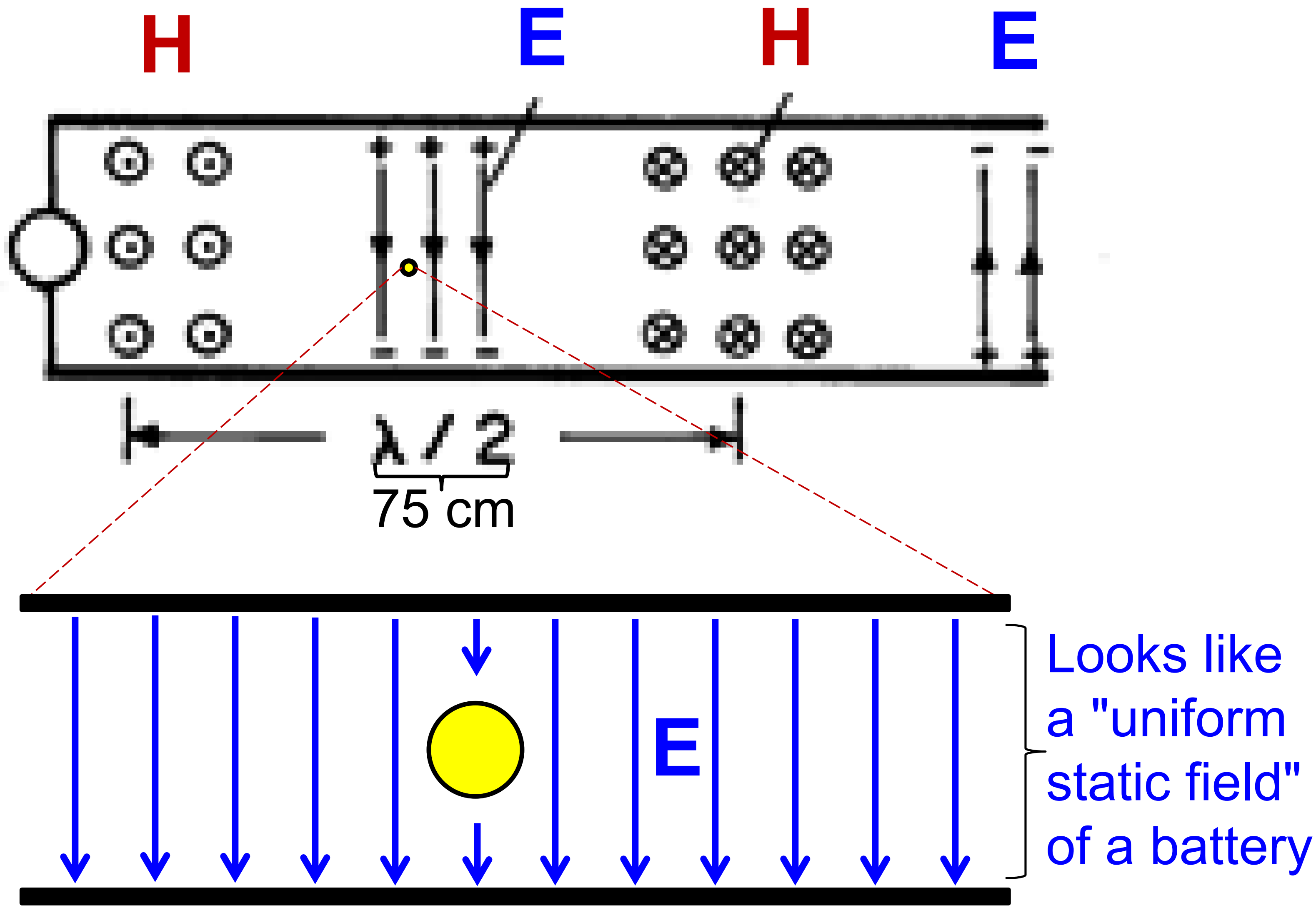




f = <u>200 MHz</u>; $\lambda = [c/f] = \underline{\lambda = 1.5\ \text{m}}$

What E field does a 7 µm diam cell “see” in a 1.5 m wavelength EM wave?





$\lambda_{\text{EM-wave}} >>> L_{\text{cell}}$

low enuf freq = long $\underline{\lambda}$ $\Rightarrow$ **E**lectro **Q**uasi **S**tatic Limit

PSet 4, P1 (Show that……)

From EM Waves to Quasistatics

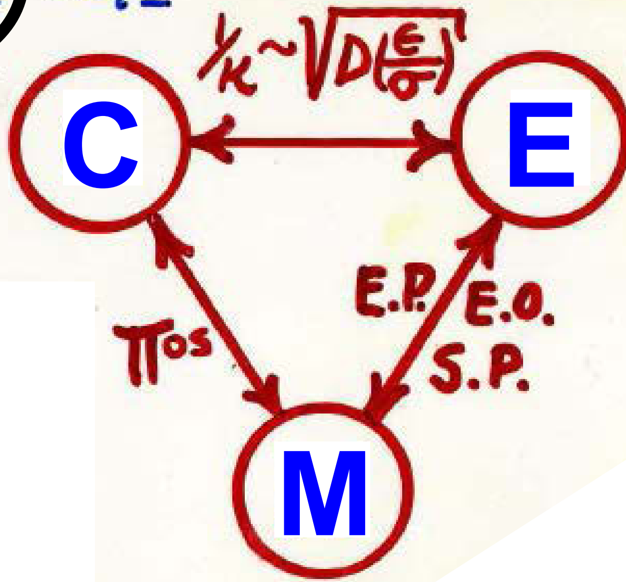
- Show that this quasistatic limit corresponds to the case where the **wavelength λ** of the EM wave **>> characteristic length L of the system** (e.g., “L” of a tissue, cell, etc.)…. …..use scaling analysis with Maxwell’s eqns.
- RESULT: **($\nabla \times \boldsymbol{E} \approx 0$) can be replaced by ($E = -\nabla\Phi$) and don’t worry about EM Waves!**

FFF: Complete Description of Coupled Transport and Biomolecular Interactions

$$\underline{N}_i = -D_i \nabla c_i + \frac{z_i}{|z_i|} u_i c_i \underline{E} + c_i \underline{v}$$

$$\frac{\partial c_i}{\partial t} = -\nabla \cdot \underline{N}_i + R_{vi}$$

Diffusion-Reaction



$$\nabla \cdot \epsilon \underline{E} = \rho_e = \sum_i z_i F c_i$$

$$(\underline{E} = -\nabla \Phi)$$

$$\nabla \cdot \underline{J} = -\frac{\partial \rho_e}{\partial t}$$

$$\underline{J} = \sum_i z_i F \underline{N}_i$$

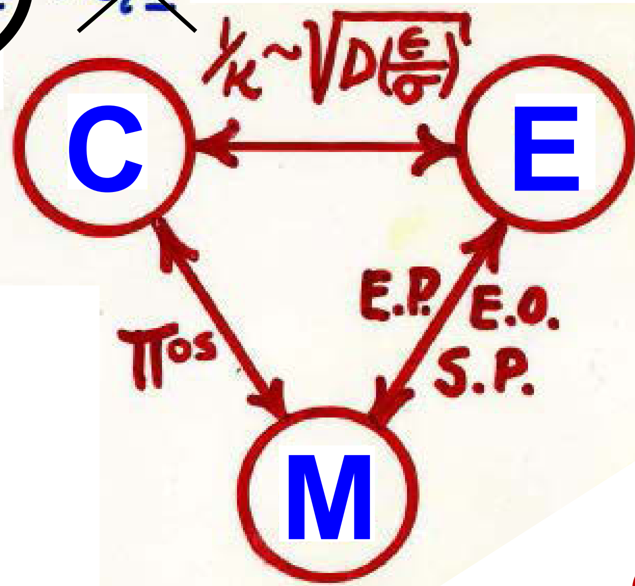
Quasi-Static (EQS)

FFF: Complete Description of Coupled Transport and Biomolecular Interactions

$$\underline{N}_i = -D_i \nabla c_i + \frac{z_i}{|z_i|} u_i c_i \underline{E} + c_i \underline{U}$$

$$\frac{\partial c_i}{\partial t} = -\nabla \cdot \underline{N}_i + R_{vi}$$

Diffusion-Reaction



$$\nabla \cdot \epsilon \underline{E} = \rho_e = \sum_i z_i F c_i$$

$$(\underline{E} = -\nabla \Phi)$$

$$\nabla \cdot \underline{J} = -\frac{\partial \rho_e}{\partial t}$$

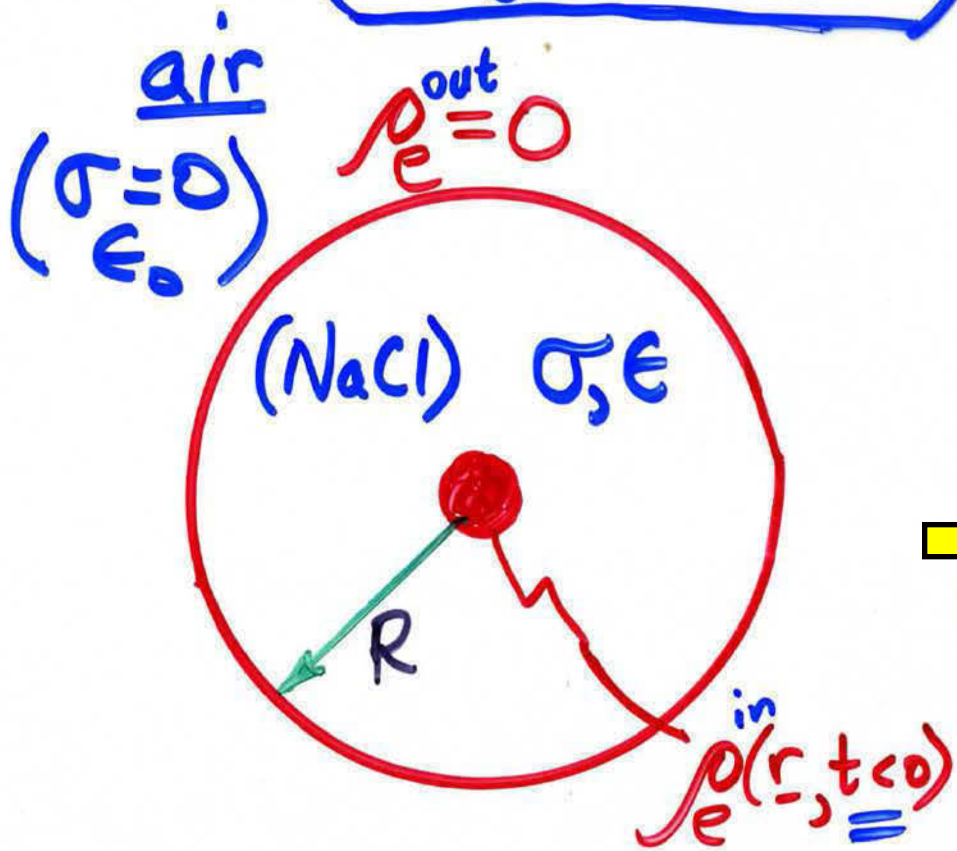
$$\underline{J} = \sum_i z_i F \underline{N}_i$$

Quasi-Static

- **Assume for now: all current is "Ohmic"**
  (neglect diffusion and convection)

$$\sigma = \sum (|z_i| F u_i c_i) \quad \Leftarrow \quad \boldsymbol{J} = \sigma \boldsymbol{E}$$

Charge Relaxation



At $t=0$, turn off ρ_e^{in}

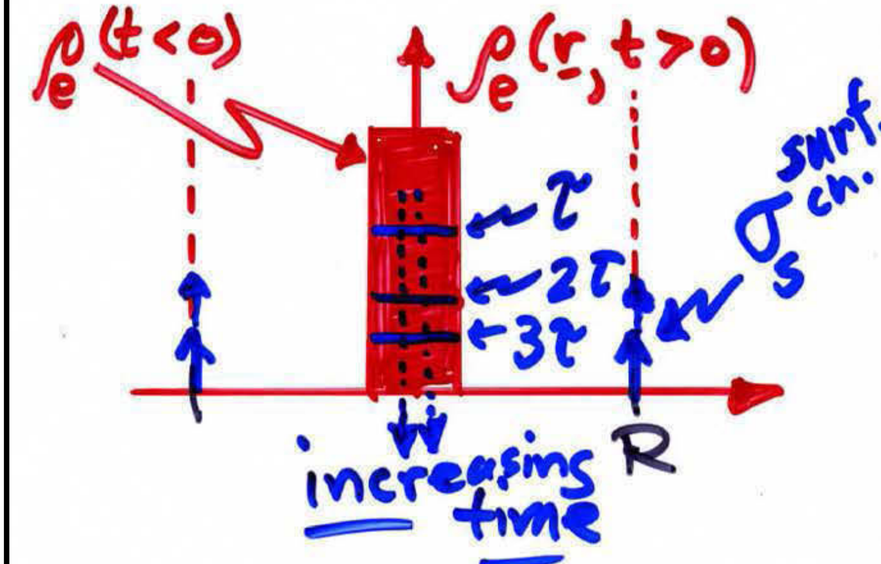
$$\nabla\cdot\underline{J} = -(\partial\rho_e/\partial t)$$

$$\frac{\partial\rho_e}{\partial t} + \frac{\sigma}{\epsilon}\underbrace{\nabla\cdot\epsilon\underline{E}}_{\rho_e} = 0$$

$$\rho_e = [\rho_e(\underline{r}, t=0)]\, e^{-t/\tau_{ch.\,rel.}}$$

$$\tau_{ch.\,rel.} \sim (\epsilon/\sigma)$$

Charges ρ_e migrate to insulating interfaces

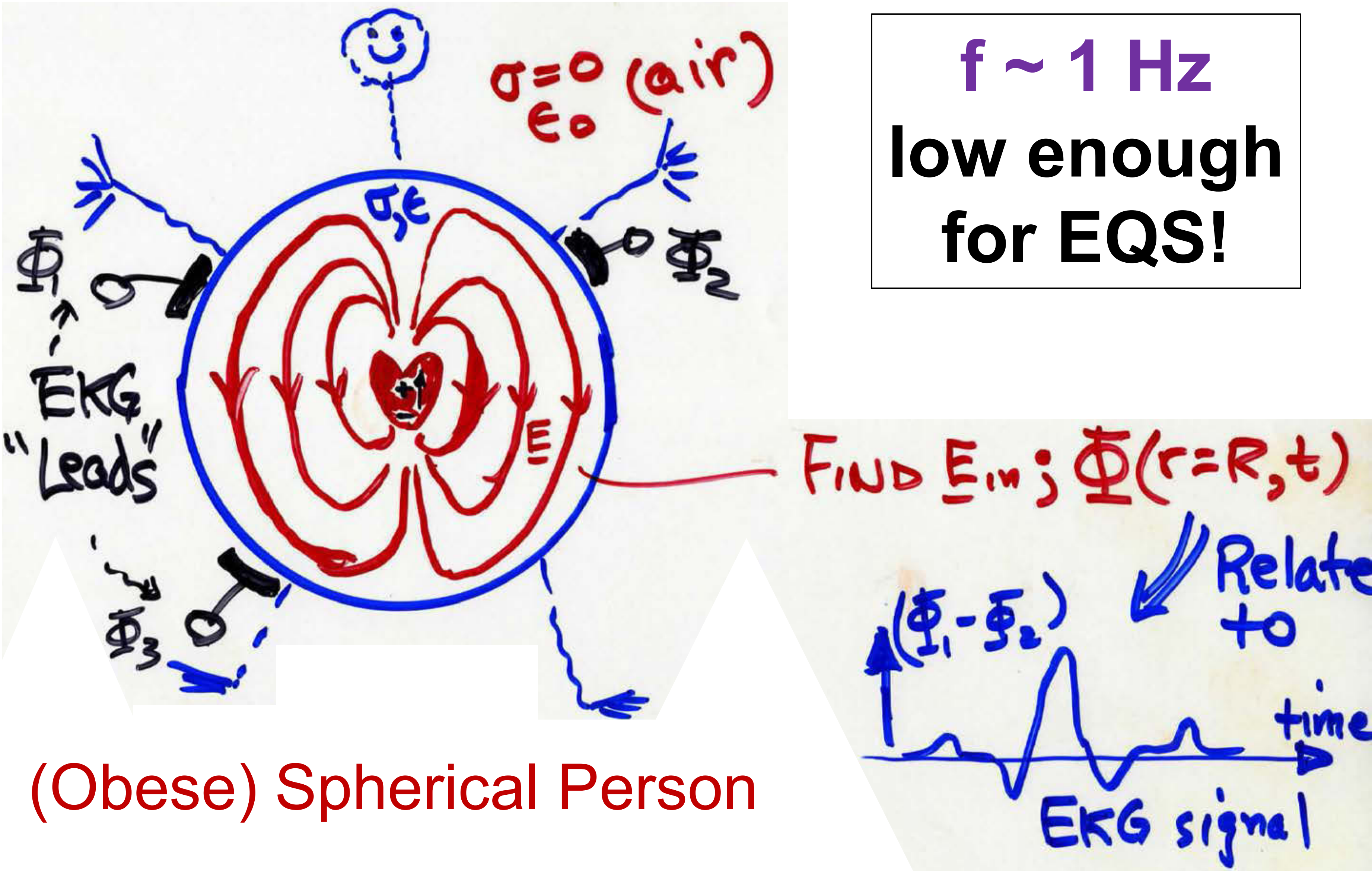


EQS subset of Maxwell's Eqns

LAW	Boundary Cond.
(1) $\nabla \cdot \epsilon \underline{E}(\underline{r},t) = \rho_e(\underline{r},t)$ **Gauss**	(1') $\underline{n} \cdot (\epsilon_o \underline{E}_1 - \epsilon \underline{E}_2) = \sigma_s(t)$
(2) $\nabla \times \underline{E} = 0$ $\Rightarrow E = -\nabla\Phi$ **Faraday**	(2') $\underline{n} \times (\underline{E}_1 - \underline{E}_2) = 0$ $\Leftrightarrow \Phi_1 = \Phi_2$ * (E_{tan} continuous)
(3) $\nabla \cdot \underline{J}(\underline{r},t) = -\frac{\partial}{\partial t}\rho_e(\underline{r},t)$ **Cons. of Charge**	(3') $\underline{n} \cdot (\underline{J}_1 - \underline{J}_2) = -\frac{\partial \sigma_s(t)}{\partial t}$ $\Leftrightarrow$ * $\underline{n} \cdot (\sigma_1 \underline{E}_1 - \sigma \underline{E}_2) = -\frac{\partial \sigma_s}{\partial t}$ ($\sigma_1 \to 0$)

(4) $J = \sigma E$ (+ other) (constitutive law)

P5.1: EKG: Centric Dipole Model of the Heart



f ~ 1 Hz
low enough for EQS!

(Obese) Spherical Person

Beating Heart is still a solution of Laplace: $\nabla^2\Phi = 0$

Table 2.8 Quasistatic laws for linear media.

Electroquasistatic (EQS)
$\nabla \cdot \epsilon \boldsymbol{E} = \rho_e$
$\nabla \times \boldsymbol{E} = 0$
$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho_e}{\partial t} \simeq 0$

$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho_e}{\partial t} \simeq 0 \leftrightarrow$ "Steady" Conduction (sec 2.7.1)

(since $\tau_{\text{heart}} \ggg \tau_{\text{ch. rel.}}$)

τ_{heart}: ~1 sec

$\tau_{\text{ch. rel.}}$: ~10^{-9} sec in physiologic media

$\boldsymbol{J} = \sigma \boldsymbol{E}$

$\nabla \cdot J = \sigma \nabla \cdot E = 0$

$\nabla^2\Phi = 0$ (Laplace)

Source: Grodzinsky, Alan. Field, Forces and Flows in Biological Systems. Garland Science, 2011. [Preview with Google Books]

Beating Heart is still a solution of Laplace: $\nabla^2\Phi = 0$

Table 2.8 Quasistatic laws for linear media.

Electroquasistatic (EQS)

$\nabla \cdot \epsilon \boldsymbol{E} = \rho_e$ **= 0** inside a uniform conductor carrying current

$\nabla \times \boldsymbol{E} = 0$

$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho_e}{\partial t} \simeq 0$ ↔ "Steady" Conduction (sec 2.7.1)

(since $\tau_{heart} >>> \tau_{ch.\ rel.}$)

τ_{heart}: **~1 sec**

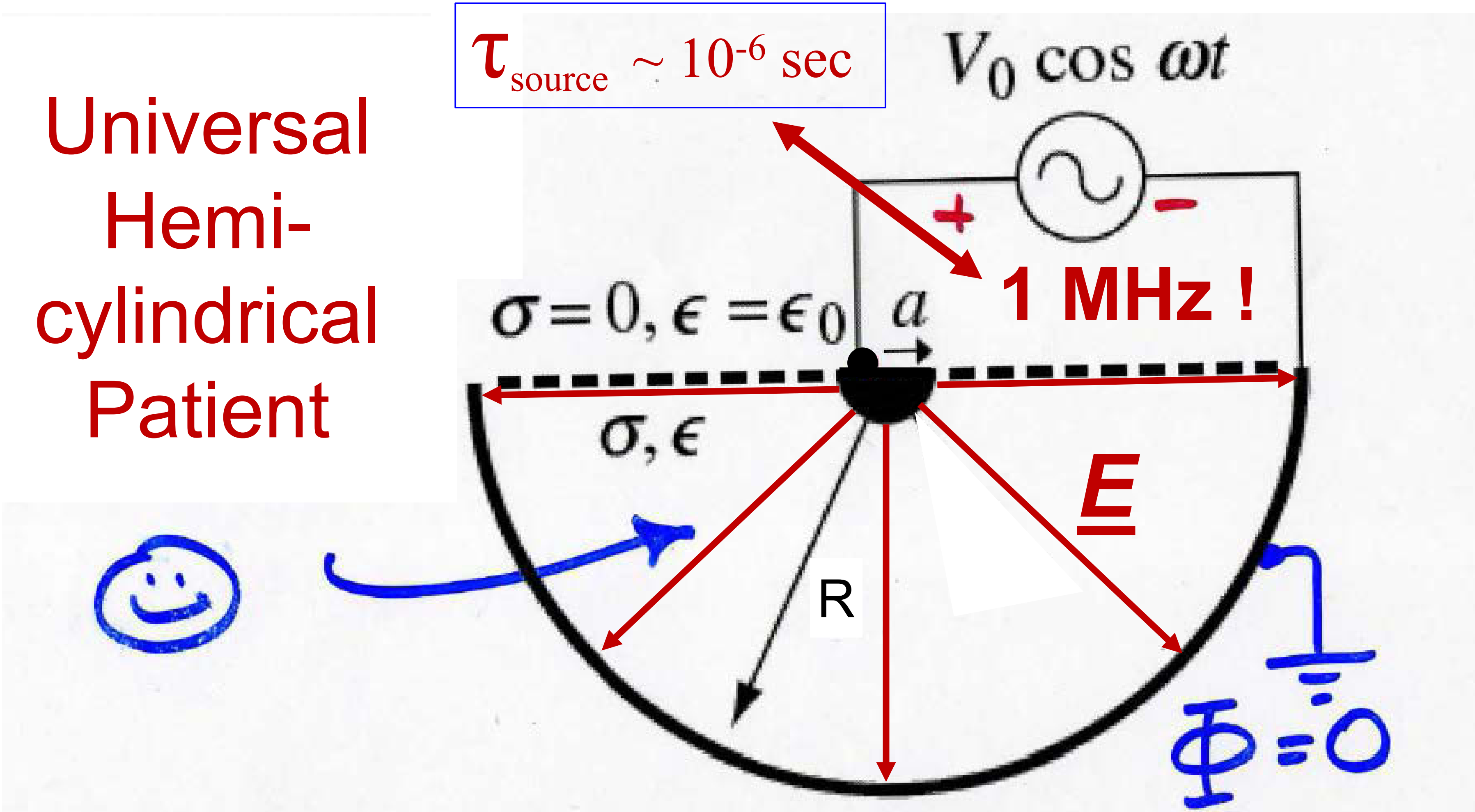
$\tau_{ch.\ rel.}$: **~10^{-9} sec in physiologic media**

$\boldsymbol{J} = \sigma \boldsymbol{E}$

$\nabla \cdot J = \sigma \nabla \cdot E = 0$

$\nabla^2\Phi = 0$

Electrosurgery: Cutting and Coagulation

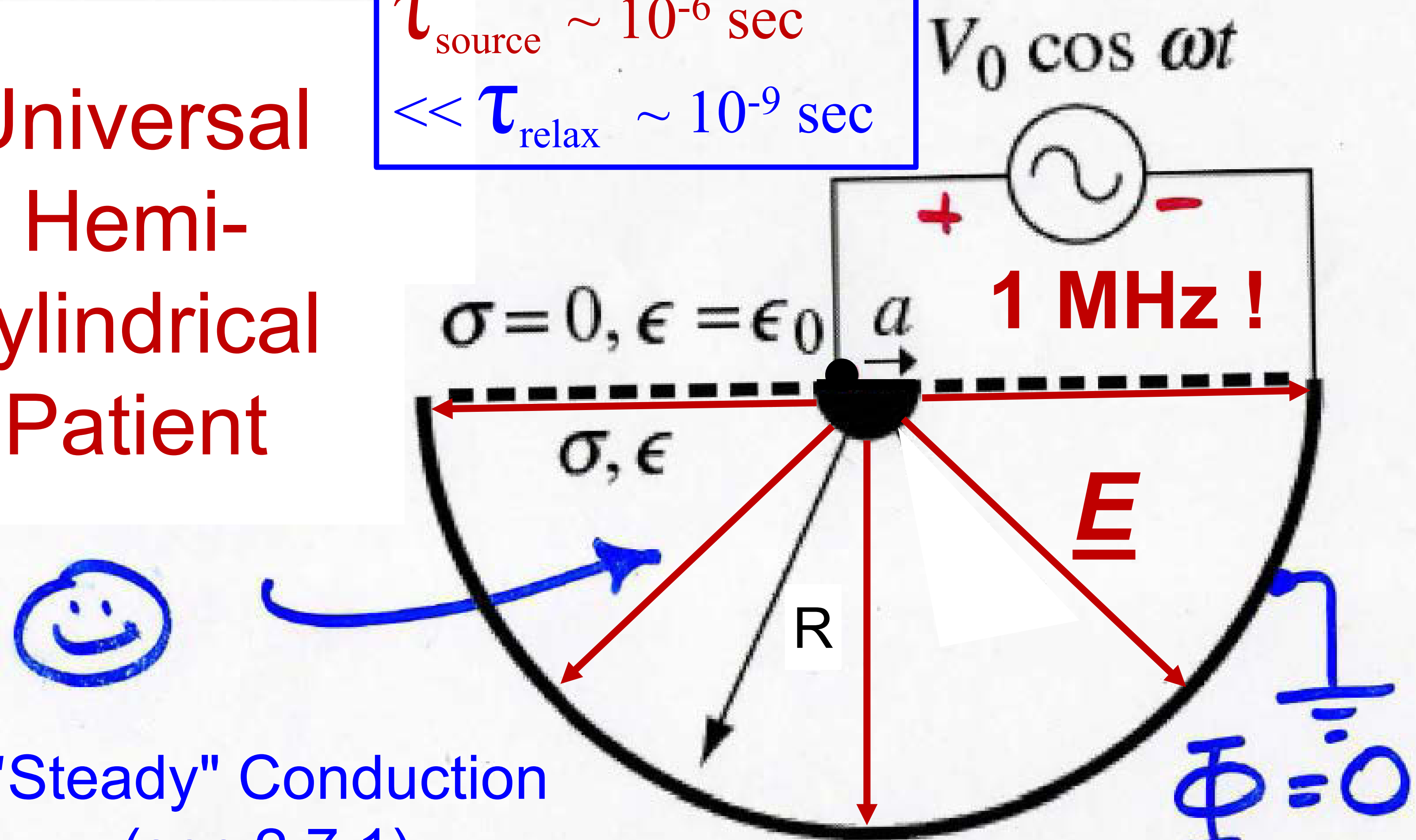


Prob. 2.7 in Text

Electrosurgery: Cutting and Coagulation

$\tau_{source} \sim 10^{-6}$ sec
$<< \tau_{relax} \sim 10^{-9}$ sec

Universal Hemi-cylindrical Patient



"Steady" Conduction (sec 2.7.1)

$\nabla^2 \Phi = 0$ (Laplace)

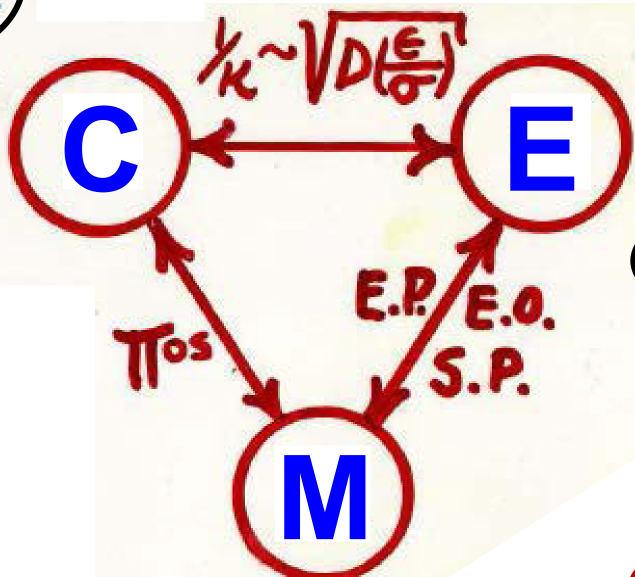
Power $\propto$ $\boldsymbol{J} \cdot \boldsymbol{E} \sim |\sigma E^2|$

FFF: Complete Description of Coupled Transport and Biomolecular Interactions

$$\underline{N}_i = -D_i \nabla c_i + \frac{z_i}{|z_i|} u_i c_i \underline{E}$$

$$\frac{\partial c_i}{\partial t} = -\nabla \cdot \underline{N}_i + R_{vi}$$

Diffusion-Reaction



$$\nabla \cdot \epsilon \underline{E} = \rho_e = \sum_i z_i F c_i$$

$$(\underline{E} = -\nabla \Phi)$$

$$\nabla \cdot \underline{J} = -\frac{\partial \rho_e}{\partial t} \simeq 0$$

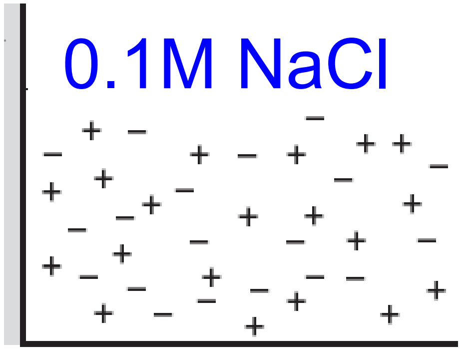
$$\underline{J} = \sum_i z_i F \underline{N}_i$$

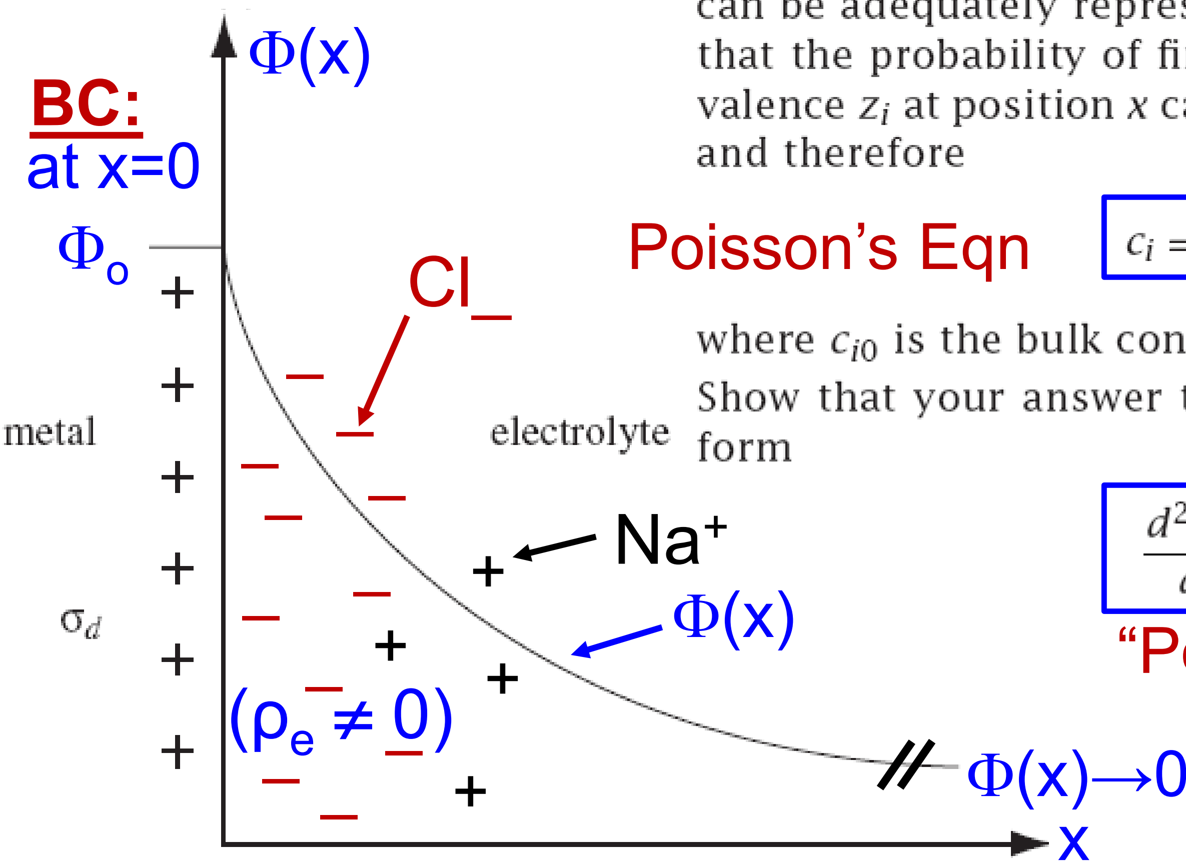
Quasi-Static (EQS)

$$\tau_{source} >> \tau_{relax}$$

$$\nabla^2 \Phi = 0 \quad \text{(Laplace)}$$

PSet 4, Prob 2





In Figure 2.24, we picture an idealized metal electrode/electrolyte interface where the metal is known to have a net surface charge σ_d at $x = 0$. This leads to a net space charge of mobile ions in the adjacent electrolyte phase. We wish to find the *equilibrium* potential and space charge distribution in the electrolyte.

(**a**) For the one-dimensional model of Figure 2.24, write Poisson's equation for the electrolyte region $x \geq 0$.

Further, assume that the distribution of all mobile ions can be adequately represented by Boltzmann statistics, so that the probability of finding a given ion of species i and valence z_i at position x can be written as $\exp[-z_i F \Phi(x)/RT]$, and therefore

Poisson's Eqn

$$c_i = c_{i0} e^{-z_i F \Phi(x)/RT} \tag{2.127}$$

where c_{i0} is the bulk concentration of the ith species.

Show that your answer to part (a) reduces to the limiting form

$$\frac{d^2\Phi(x)}{dx^2} = \kappa^2 \Phi(x) \tag{2.128}$$

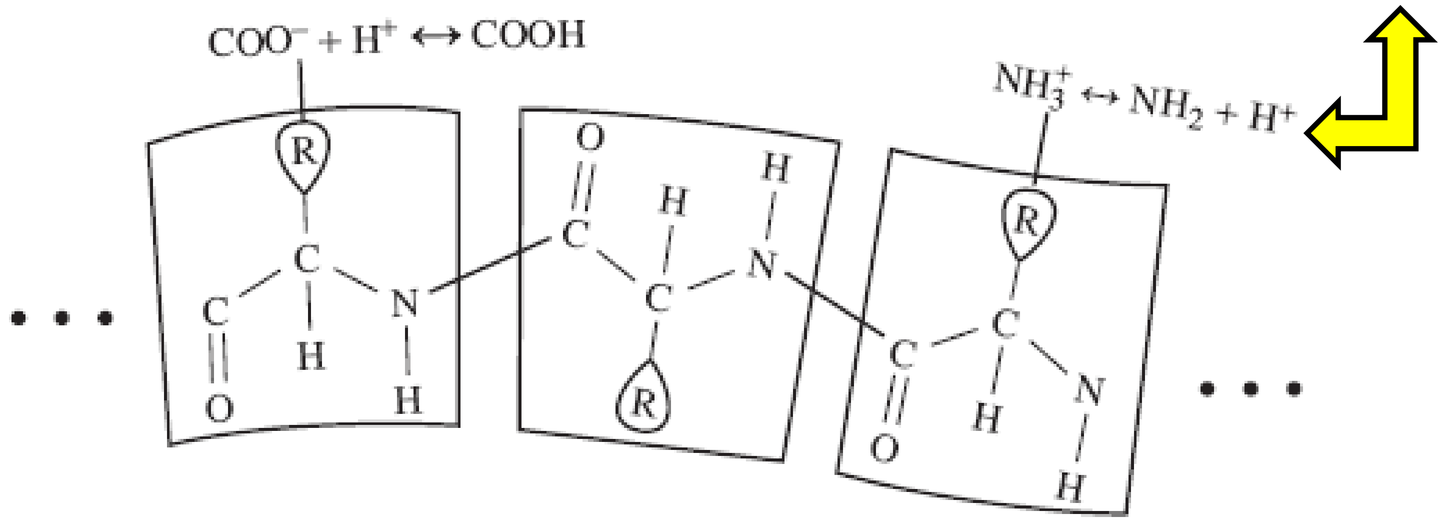
"Poisson-Boltzmann Eqn" (linearized)

….Find Φ(x)

Charge of Amino Acid Residues

ASP (D)   pK ~ 3.9
GLU (E)   pK ~ 4.3

ARG (R)   pK ~ 12.5
LYS (K)   pK ~ 10.5



$$-COOH \rightleftharpoons COO^- + H^+$$

$$-NH_3^+ \rightleftharpoons NH_2 + H^+$$

Dissociation Reactions Depend on pH (& ionic strength)

p. 18 of Text