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Molecular, Cellular & Tissue Biomechanics Spring 2002

Problem Set #4

Issued: Wed., 3/13/02
Due: Wed., 3/20/02

READING ASSIGNMENT: Sections 8.5 and 8.7 in the Grodzinsky-Chapter 8 notes.

Problem 1 Oscillatory Compression of Poroelastic Tissue

This problem is modeled after the poroelastic stress relaxation response derived in the previous Problem Set [3.2, parts (c) and (e)]. Using the same confined compression geometry of Problem 3.2, we now wish to find an expression for the displacement $\hat{u}(z,\omega)$ that results from an applied "sinusoidal steady state" displacement $u(z=0,t) = u_o\, cos\omega t$ at the surface $z = 0$.

(a) From the poroelastic diffusion equation for $u(z,t)$, let the solution $u(z,t)$ have the form

$$u(z,t) = ReAe^{j\alpha(L-z)}e^{j\omega t}$$

which represents a "diffusion wave" at frequency ω decaying in the z-direction with the space constant α. Inserting this form into the diffusion equation along with $(\partial/\partial t \to j\omega)$, show that

$$\alpha = \pm(1-j)\sqrt{\omega/2Hk}$$

(b) Since the complex diffusion equation is second order in z, the solution is the superposition of two solutions corresponding to $+\alpha$ and $-\alpha$. Using the boundary conditions on $\hat{u}(z,\omega)$ at z=0 and z=L, show that the solution has the form:

$$\hat{u}(z,\omega) = \frac{u_o sinh\gamma(L-z)}{sinh\gamma L}$$

Find γ in terms of material properties and frequency, and interpret the poroelastic "penetration depth" γ by sketching your solution at an instant in time, but identifying the "envelope" $sinh\gamma L$.

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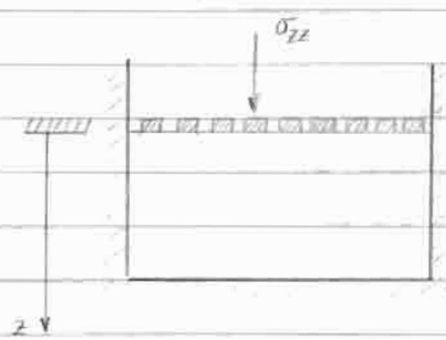
BEH 410 PROBLEM SET 4 due 3/20/2002

Problem 1. Oscillatory compression of poroelastic tissue



Poroelastic stress relaxation response under confined compression conditions; when a "sinusoidal steady state" displacement $u(z=0, t) = u_0 \cos \omega t$ is applied.
The solution $u(z,t) = \mathrm{Re}\left[\hat{u}(z,\omega)\exp(j\omega t)\right]$ is solution of

$$\frac{\partial u}{\partial t} = Hk \frac{\partial^2 u}{\partial z^2} \qquad (1)$$

where H = confined compression coefficient and k = Darcy's hydraulic permeability.

a) Supposing that $\hat{u} = A \exp\left(j\alpha(L-z)\right)$, and knowing that $\hat{u}$ is solution of

$$j\omega \hat{u} = Hk \frac{\partial^2 \hat{u}}{\partial z^2}, \qquad (2)$$

we get $\frac{\partial^2 \hat{u}}{\partial z^2} = -\alpha^2 \hat{u}$, and (2) is satisfied for

$$\alpha^2 = -j \frac{\omega}{Hk}$$

$$\boxed{\alpha = \pm (1-j)\sqrt{\frac{\omega}{2Hk}}}$$

✓

Indeed, $\left(\frac{1-j}{\sqrt{2}}\right)^2 = \frac{1^2 + j^2 - 2j}{2} = -j$

b) Since the complex diffusion equation (2) is second order in z, the solution is the superposition of two solutions corresponding to α and $-\alpha$ respectively.

$$\hat{u}(z,\omega) = A_+ \exp\left\{ j(1-j)\sqrt{\frac{\omega}{2Hk}}(L-z) \right\} + A_- \exp\left\{ -j(1-j)\sqrt{\frac{\omega}{2Hk}}(L-z) \right\}$$

We have furthermore $\hat{u}(z=L,\omega) = 0$ (3)

$\hat{u}(z=0,\omega) = u_0$ (4)

From (3) we get $A_+ = -A_-$

From (4) we get $u_0 = A_+ \{\exp(j\alpha L) - \exp(-j\alpha L)\}$

$= 2A_+ \sinh \alpha L$

and $\hat{u}(z,\omega) = 2A_+ \sinh \alpha(L-z)$

Eventually $\boxed{\hat{u}(z,\omega) = \frac{u_0 \sinh \gamma(L-z)}{\sinh \gamma L}}$ ✓ where $\underline{\gamma = \alpha}$ complex

The "penetration depth" is a characteristic length.

$$\boxed{\delta = \frac{1}{\mathrm{Re}\,\alpha} = \frac{1}{\mathrm{Re}\,\gamma} = \sqrt{\frac{2Hk}{\omega}}}$$