16.901: Homework # 9
Solution

In the lecture notes on the method of weighted residuals, we solved a one-dimensional heat diffusion problem using a combination of a quadratic and cubic polynomial. In this homework, you are again to apply the method of weighted residuals to the same problem as in the lecture notes, but we will use a different set of functions. As before, we will use two terms,

$$\tilde{T}(x) = 100 + \sum_{i=1}^{2} a_i \phi_i(x),$$

This time, we will use sinusoidal functions, specifically,

$$\begin{aligned} \phi_1(x) &= \cos\left(\frac{\pi}{2}x\right), \\ \phi_2(x) &= \sin(\pi x). \end{aligned}$$

Note that these functions satisfy the boundary conditions that $\phi_i(\pm 1) = 0$.

1. Using the (Galerkin) method of weighted residuals, determine the coefficients a_1 and a_2.

   Solution: The residual for this problem is given by,

$$R(\tilde{T}, x) \equiv \tilde{T}_{xx} + q.$$

   For our functions,

$$\begin{aligned} (\phi_1)_{xx} &= -\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}x\right), \\ (\phi_2)_{xx} &= -\pi^2 \sin(\pi x), \\ \Rightarrow \tilde{T}_{xx} &= -a_1\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}x\right) - a_2\pi^2\sin(\pi x), \\ \Rightarrow R(\tilde{T}, x) &= -a_1\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}x\right) - a_2\pi^2\sin(\pi x) + 50e^x. \end{aligned}$$

   Using a Galerkin method, the weighted residuals are,

$$\begin{aligned} R_1(\tilde{T}) &= \int_{-1}^{1} w_1(x)\, R(\tilde{T}, x)\, dx, \\ &= \int_{-1}^{1} \cos\left(\frac{\pi}{2}x\right)\left[-a_1\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}x\right) - a_2\pi^2\sin(\pi x) + 50e^x\right] dx, \\ R_2(\tilde{T}) &= \int_{-1}^{1} w_2(x)\, R(\tilde{T}, x)\, dx, \\ &= \int_{-1}^{1} \sin(\pi x)\left[-a_1\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}x\right) - a_2\pi^2\sin(\pi x) + 50e^x\right] dx. \end{aligned}$$

   Then, the following integrals are used,

$$\int_{-1}^{1} \cos^2\left(\frac{\pi}{2}x\right) dx = 1, \quad \int_{-1}^{1} \sin^2(\pi x)\, dx = 1, \quad \int_{-1}^{1} \cos\left(\frac{\pi}{2}x\right)\sin(\pi x)\, dx = 0,$$

$$\int_{-1}^{1} \cos\left(\frac{\pi}{2}x\right) e^x\, dx = \frac{\pi}{1+\pi^2/4}\cosh 1, \quad \int_{-1}^{1} \sin(\pi x)\, e^x\, dx = \frac{2\pi}{1+\pi^2}\sinh 1$$

giving,

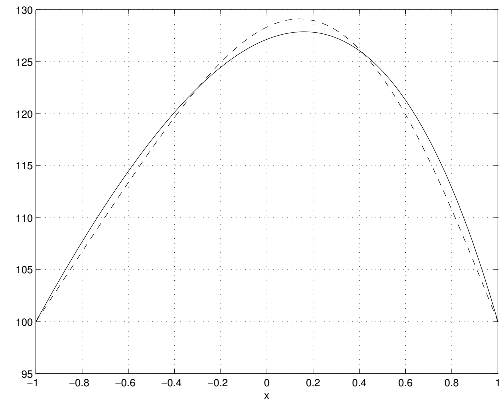
$$
\begin{aligned}
R_1(\tilde{T}) &= -a_1\frac{\pi^2}{4} + \frac{50\pi}{1+\pi^2/4}\cosh 1, \\
R_2(\tilde{T}) &= -a_2\pi^2 + \frac{100\pi}{1+\pi^2}\sinh 1.
\end{aligned}
$$

Setting these weighted residuals to zeros gives,

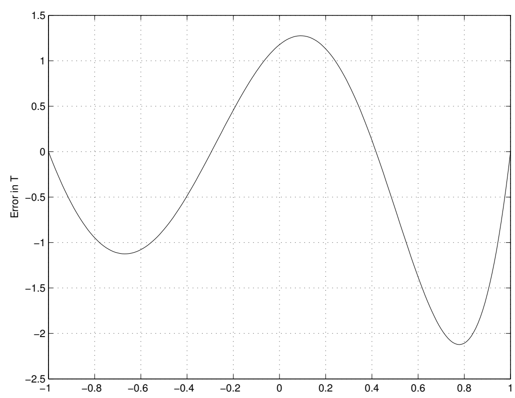
$$
\begin{aligned}
a_1 &= \frac{1}{\pi}\frac{200}{1+\pi^2/4}\cosh 1 = 28.331, \\
a_2 &= \frac{1}{\pi}\frac{100}{1+\pi^2}\sinh 1 = 3.442.
\end{aligned}
$$

2. Plot the approximate solution and the exact solution on the same graph. Also, plot the error and the residual. Compare the quality of this sinusoidal approximation to the polynomial approximation (using the method of weighted residuals) applied in the lecture notes.

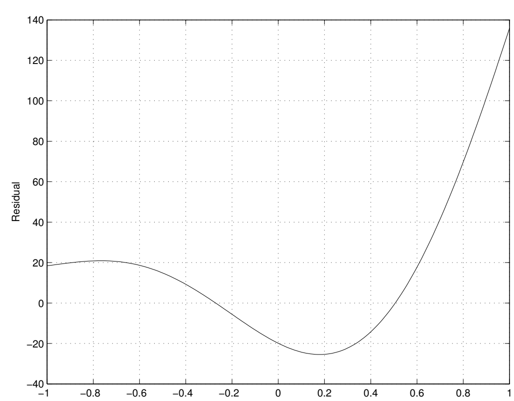
   Solution: The results are shown in Figure 1. The approximation is not quite as good as that achieved with the polynomial approximation used in the lecture notes, but still reasonable.



(a) Comparison of T (solid) and $\tilde{T}$ (dashed)



(b) Error, $\tilde{T} - T$



(c) Residual, $R(\tilde{T}, x)$

Figure 1: Results for method of weight residuals using sinusoidal functions.