16.901: Homework # 6
Solution

Implement the fourth-order Runge-Kutta algorithm to solve the problem

$$u_t = -u^2 \quad \text{with} \quad u(0) = 1.$$

Applying this method for timesteps of $\Delta t = 0.01$, 0.02, and 0.04, plot the error from $t = 0$ to 10. Based on these results, show that the actual error is converging at fourth-order with respect to Δt.

Solution: In the past, to determine the rate of convergence from numerical results, we determined the factor by which the error was reduced at a specific time t when Δt was decreased. That is acceptable, but even better is to quantify the error reduction for all t. To do this, the error is plotted by normalizing by the error for the smallest Δt results at the same time. The attached Matlab script does that. The results are shown in Figure 1. As can be seen from this plot, the error is a factor of 16 and 256 larger than the $\Delta t = 0.01$ results for corresponding doubling and quadrupling of Δt. This agrees with our expectation that the error be fourth-order, error $\approx \Delta t^4$ since we would expect the error to be larger by a factor of $2^4 = 16$ and $4^4 = 256$.

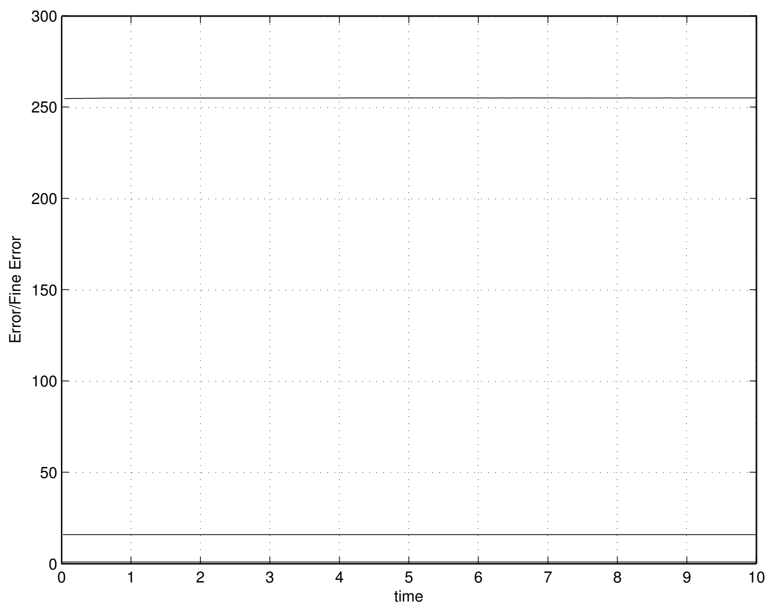

Figure 1: Error for fourth-order Runge-Kutta approximation of $u_t = -u^2$ with $u(0) = 1$ and $\Delta t = 0.01$, 0.02, and 0.04. Note: errors are normalized by the errors when using the smallest $\Delta t = 0.01$.