

16.901 Homework #4  
Due Date: March 17, 2pm

### Problem #1: Eigenvalues of the 1-D Euler Equations

The Euler equations for one-dimensional flow are,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, \quad (1)$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} = -\frac{\partial p}{\partial x}, \quad (2)$$

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial(\rho u E)}{\partial x} = -\frac{\partial(\rho u)}{\partial x}, \quad (3)$$

where  $\rho$  is the density,  $u$  is the flow velocity,  $p$  is the pressure, and  $E$  is the total energy. When solving these equations using a numerical approach, it is common to re-write these Euler equations in the conservative form using the conservative state vectors,

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{pmatrix}, \quad (4)$$

where  $H$  is the total enthalpy and is related to the previous quantities by,

$$H = E + \frac{p}{\rho}.$$

Note, the total energy ( $E$ ) and total enthalpy ( $H$ ) are related to the energy ( $e$ ) and enthalpy ( $h$ ) by the following,

$$E = e + \frac{1}{2}u^2$$

$$H = h + \frac{1}{2}u^2$$

To close this set of equations, we need thermodynamic state relationships. We will assume the working fluid is air and can be assumed to be an ideal, perfect gas. In this case, the state equations for the pressure and energy are,

$$p = \rho R T, \quad (5)$$

$$e = c_v T, \quad (6)$$

where  $R$  is the gas constant,  $T$  is the temperature, and  $c_v$  is the specific heat at constant volume. Another useful thermodynamic quantity is the specific heat at constant pressure,  $c_p$ , which is related to  $R$  and  $c_v$  by,

$$c_p - c_v = R.$$

Also, the ratio of specific heats  $\gamma = c_p/c_v$ . Finally, for a perfect gas, the speed of sound,  $c$  is given by,

$$c = \sqrt{\frac{\gamma p}{\rho}}.$$

1. By manipulating the various definitions and thermodynamic relationships, show that the pressure is given by,

$$p = (\gamma - 1) \left( \rho E - \frac{1}{2} \rho u^2 \right).$$

2. In this step, we will derive the linearized form of the 1-D Euler equations. Specifically, consider the state vector to be composed of a constant and perturbation,

$$U(x, t) = \bar{U} + \epsilon \tilde{U}(x, t), \quad (7)$$

where  $\bar{U}$  is the constant state vector (i.e. it does not depend on either  $x$  or  $t$ ), and  $\tilde{U}$  is the perturbation state which does depend on  $x$  and  $t$ . Introducing Equation (7) into Equation (4), show that the equation governing  $\tilde{U}$  (for  $\epsilon \ll 1$ ) is,

$$\frac{\partial \tilde{U}}{\partial t} + A(\bar{U}) \frac{\partial \tilde{U}}{\partial x} = 0, \quad (8)$$

where,

$$A(\bar{U}) = \left. \frac{\partial F}{\partial U} \right|_{U=\bar{U}}.$$

3. Show that the matrix  $A$  for the Euler equations is,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma - 3)u^2 & (3 - \gamma)u & \gamma - 1 \\ (\gamma - 1)u^3 - \gamma uE & -\frac{3}{2}(\gamma - 1)u^2 + \gamma E & \gamma u \end{pmatrix}.$$

4. Prove that solutions to the linearized Euler equations exist which have the form,  $\tilde{U}(x, t) = \tilde{U}(x - \lambda t)$  where  $\lambda$  are the eigenvalues of the  $A$  matrix. Calculate the  $\lambda$  for the 1-D Euler equations.

## Problem #2: Upwind Differencing and Numerical Dissipation

1. Consider the 1-D partial differential equation,

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = 0.$$

To apply upwind-differencing (i.e. one-sided) to this equation in the general case where the velocity  $u$  can be positive or negative, the following approximations are used,

$$u_i \left. \frac{\partial T}{\partial x} \right|_i = \begin{cases} u_i \delta_x^+ T_i & \text{if } u_i < 0 \\ u_i \delta_x^- T_i & \text{if } u_i > 0 \end{cases}$$

Prove that this direction-dependent differencing scheme can be written as

$$u_i \left. \frac{\partial T}{\partial x} \right|_i = u_i \delta_{2x} T_i - \frac{1}{2} |u_i| \Delta x \delta_x^2 T_i, \quad (9)$$

where  $\delta_{2x}$  and  $\delta_x^2$  are standard central-difference operators for 1st and 2nd derivatives.

2. In the early development of Computational Fluid Dynamics (CFD) algorithms to solve inviscid flows, a common approach was to add numerical (non-physical) dissipation to improve the stability of the discretizations. Specifically, consider the modified equation,

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = k_{num} \frac{\partial^2 T}{\partial x^2},$$

where  $k_{num}$  is a numerical viscosity coefficient. If the  $x$ -derivatives are all approximated with standard central-difference approximations, what is the value of  $k_{num}$  such that the resulting finite difference method will be identical to the previous upwind discretization given by Equation (9).

3. When solving viscous flows, the possibility exists that the numerical algorithm could be stabilized by the presence of the physical dissipation. As a simplified model equation, suppose we were interested in solving

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial x^2}.$$

The Reynolds number is used to determine the relative importance of inviscid to viscous effects. For an airfoil problem, the Reynolds number is defined as,

$$\text{Re} = \frac{uc}{\nu},$$

where  $c$  is the chord length of the airfoil. A typical Reynolds number for an airfoil on a commercial, transonic aircraft would be around  $10^7$ . For this situation, what is the  $\Delta x/c$  required so that the physical dissipation would be equal to the numerical dissipation produced by an upwind approximation? Based on this result, is it reasonable to consider stabilizing a numerical approximation using only physical dissipation?

### Problem #3: Matrix Stability Analysis for the Convection Equation

In this problem, we will analysis the eigenvalue stability of a central-difference discretization of the convection equation,

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = 0.$$

Thus, the semi-discrete scheme obtained with a central-difference approximation is,

$$\frac{\partial T_j}{\partial t} + u \delta_{2x} T_j = 0.$$

We will assume the problem is periodic of a length  $L$  and that the grid is constructed of  $J$  equally-spaced points.

1. Show that the resulting semi-discrete equation (i.e. discretized only in space, not in time) for this approximation has the following form,

$$\frac{d\hat{T}}{dt} + A\hat{T} = 0,$$

where  $\hat{T}$  is the vector of unknown  $T$  values at each node,

$$\hat{T}(t) = [T_1(t), T_2(t), T_3(t), \dots, T_J(t)],$$

and  $A$  is a matrix of the form,

$$A = \begin{pmatrix} b & c & 0 & 0 & 0 & a \\ a & b & c & 0 & 0 & 0 \\ 0 & a & b & c & 0 & 0 \\ 0 & 0 & a & b & c & 0 \\ 0 & 0 & 0 & a & b & c \\ c & 0 & 0 & 0 & a & b \end{pmatrix}.$$

Specifically, determine the values of  $a$ ,  $b$ , and  $c$  for this discretization.

2. The eigenvalues of the  $A$  are given by,

$$\lambda_j = b + (a + c) \cos\left(\frac{2\pi j}{J}\right) - i(a - c) \sin\left(\frac{2\pi j}{J}\right)$$

for  $j = 0, 1, 2, \dots, J - 1$ . Find the eigenvalues for this central-difference discretization. Plot the eigenvalues in the complex plane.

3. What is the maximum time-step which is stable for this method when using a Forward Euler time discretization? Note: do this graphically by overlaying the stability region for Forward Euler on the eigenvalues (scaled by  $\Delta t$ ). Use  $J = 20$ . Include a sample plot of the eigenvalues overlayed with the Forward Euler stability region.
4. Suppose a 4-stage Runge-Kutta scheme is used to integrate in time. What is the maximum time-step in this case? Note: do this graphically again. Use  $J = 20$ . Include a sample plot of the eigenvalues overlayed with the 4-stage Runge-Kutta stability region.