

Solution to Homework #9 Robust Design and Reliability

Due Date: Monday, 20 July

Objectives:

- Understand basic concepts of reliability engineering
- Understand the relationship of robust design to reliability

Assignment

1) Given that for small time intervals Δt

$$\lambda(t)\Delta t = \Pr(\text{The system will fail within the period } t \text{ to } t + \Delta t \mid \text{The product survives to time } t)$$

$$R(t) = \Pr(\text{The product survives to time } t)$$

$$dF(t) = \Pr(\text{The system will fail within the period } t \text{ to } t + dt)$$

$$R(t) + F(t) = 1$$

Prove that

$$R(t) = e^{-\int_0^t \lambda(\xi) d\xi}$$

Solution to #1

Consider the definition of conditional probability

$$\Pr(A \& B) = \Pr(A|B)\Pr(B)$$

The first two equations given in the problem statement fill out the right hand side of the equation. Also, in order to fail within the time interval $t + \Delta t$, a product must survive to time t , the third equation in the problem statement completes the left hand side of the definition. Therefore

~~$$\Pr(A \& B) = \Pr(A|B)\Pr(B)$$~~ (1)

The fourth equation given in the problem statement is

$$R(t) + F(t) = 1$$

Taking the derivative and rearranging yields

~~$$\frac{dR(t)}{dt} = -\lambda(t)R(t)$$~~

Combining this with (1) gives

$$\frac{dR(t)}{R(t)} = -\lambda(t)\Delta t$$

For small time intervals Δt is equivalent to dt , so

$$\frac{dR(t)}{R(t)} = -\lambda(t)dt$$

Changing t into ξ gives

$$\frac{dR(\xi)}{R(\xi)} = -\lambda(\xi)d\xi$$

Integrating both sides from 0 to t gives

$$\int_0^t \frac{dR(\xi)}{R(\xi)} = -\int_0^t \lambda(\xi)d\xi \quad \ln(R(t)) - \ln(R(0)) = -\int_0^t \lambda(\xi)d\xi$$

Since $R(0)$ is always 1 and $\ln(1)$ is zero

$$R(t) = e^{-\int_0^t \lambda(\xi)d\xi}$$

Which is the result we wanted to prove.

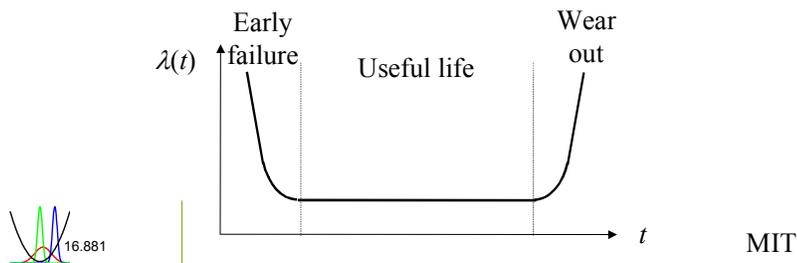
2) What form will the reliability curve $R(t)$ take if the early failure region is removed through “burn-in” and the wear-out region is avoided by retiring the product before the wear-out period? How is this form of $R(t)$ related to a Weibull distribution?

Solution to #2

The following slide appeared in the class notes

Typical Behavior

- Early failure period often removed by “burn-in”
- Wear out period sometimes avoided by retirement
- What will the reliability curve $R(t)$ look like if early failure and wear out are avoided?



If the early failure period and the wear out period are removed, then $\lambda(t)$ is constant.
From problem #1 in this homework

$$R(t) = e^{-\lambda t}$$

Since $\lambda(t)$ is constant

$$R(t) = e^{-\lambda t}$$

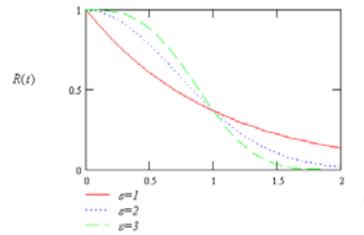
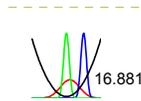
In other words, the reliability of the product degrades exponentially over its life.

From the lecture notes:

Weibull Distributions

- Common in component failure probabilities (e.g., ultimate strength of a test specimen)
- Limit of the minimum of a set of independent random variables

$$R(t) = e^{-\left(\frac{t-t_0}{\eta}\right)^s}$$



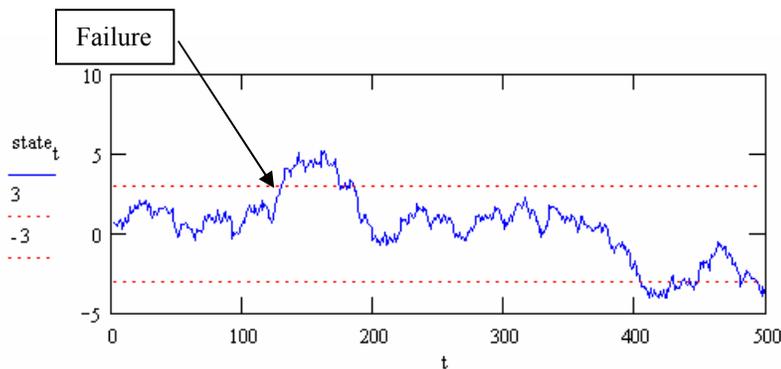
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By inspection, we can see that the distribution we just derived is the Weibull distribution with $s=1$, $t_0=0$, and $\eta=1/\lambda$.

3) If the drift in voltage of a power supply over time is approximated by a Weiner process with increments of one second and a variance of $0.1V$. If the voltage of the power supply varies from nominal by more than $3V$, it will cause the entire system to fail. What is the expected value of time to failure of the system? If the variance is cut to $0.03V$ per increment, what is the expected value of time to failure of the system? You may demonstrate your answers using closed form mathematics or simulations.

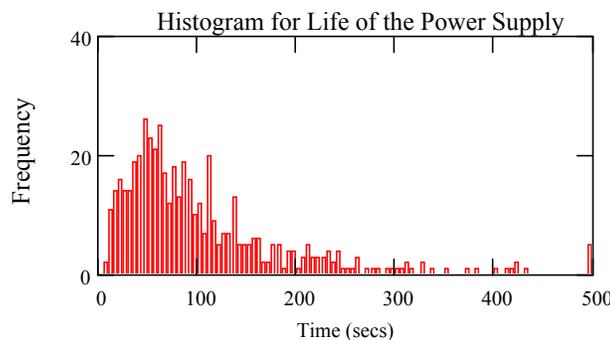
First of all, the problem statement had a typo. The units on variance of the voltage should be V^2 not V . Also, I probably should have been clearer about the meaning of the criterion for failure. I stated that “if the voltage of the power supply varies from nominal by more than $3V$, it will cause the entire system to fail” I meant that any time an individual power supply drifts away from nominal by more than $\pm 3V$ it will cause the system fail abruptly. The term “varies from nominal” may be interpreted by some as a statement concerning variance of a population of power supplies. I don’t think such a failure criterion is physically reasonable.

I ran a simulation of the process at a drift variance of $0.1V^2$. I limited the total number of time steps to 500. Here is what a typical behavior of the process looks like:



This should give you a feel for the behavior of the Weiner process. When the blue trace crosses the red dotted line, the power supply fails. Now we need to iterate many times to determine an average value of the failure time.

I set up my simulation to repeat this process 500 times and compute the mean life mean life. Below is a typical histogram of the data from a run of 500 trials. This histogram shows that the mean I computed is just a bit low. A few of the power supplies never had time to fail. Their long lives might raise the mean life a bit. Extending the time allowed for the simulation to run would raise the mean a bit.



$$\frac{f_{n-1}}{\text{no_of_reps}} = 0.998\%$$

The next step I took was to run six sets of these 500 runs of trials. The results are shown below

$$\text{mean} \begin{bmatrix} 96.37 \\ 103.11 \\ 97.202 \\ 103.154 \\ 99.678 \\ 107.826 \end{bmatrix} = 101.223 \quad \frac{1.96}{\sqrt{6}} \cdot \text{Stdev} \begin{bmatrix} 96.37 \\ 103.11 \\ 97.202 \\ 103.154 \\ 99.678 \\ 107.826 \end{bmatrix} = 3.451$$

So, the mean life is about 101 secs and I have 95% confidence that it's within about 3.5 secs of that value given the scatter in the means.

Many of you guessed that the mean life would correspond to the time at which the variance in the state of the power supply was equal to $(3V)^2$. Naturally, the variance of the state of the power supply is $n \cdot \text{SQRT}(\text{variance per step})$. This is not so unreasonable and it happens to give an answer of 90 secs, which is fairly close. However, I can't think of any sound theoretical justification for the approach.

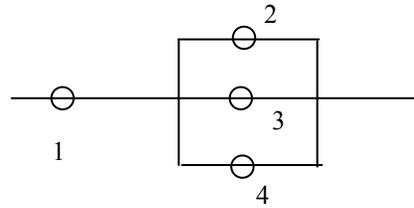
Some of you instead suggested that the mean life would correspond to the time at which the probability that the state of the power supply was within 3V of the mean equaled 0.5. This occurs when the standard deviation is 0.675 times the half tolerance width of 3V. The problem with this approach is that there are a great many opportunities for failure prior to reaching this critical value of standard deviation. Therefore, this approach substantially overestimates the time to failure of the system.

In any case, the important teaching point concerns the effect of variation on reliability. I lowered the variance to $0.03V^2$. I raised the time allowed by a factor of three to maintain a balance between the time allowed and the variance of the process. The mean time to failure tripled as expected.

$$\text{mean} \begin{bmatrix} 310.078 \\ 316.642 \\ 321.968 \\ 303.882 \\ 318.582 \\ 316.794 \end{bmatrix} = 314.658 \quad \frac{1.96}{\sqrt{6}} \cdot \text{Stdev} \begin{bmatrix} 310.078 \\ 316.642 \\ 321.968 \\ 303.882 \\ 318.582 \\ 316.794 \end{bmatrix} = 5.24$$

Please refer to the Mathcad document Weiner6.mcd on the course web page for the details of the solution.

4) The unreliability $F(t)_i$ of each element of the system below was reduced by a factor of two (through robust design of course). What was the approximate effect on system reliability $R(t)$?



This system will fail if subsystem 1 fails *or* if subsystems 2,3, *and* 4 fail.

Before the robust design effort (according to the rules from the lecture notes) the system unreliability is

$$F_{BEFORE} = 1 - (1 - F_1)(1 - F_2F_3F_4) = F_1 + F_2F_3F_4 - F_1F_2F_3F_4$$

After reducing the failure probabilities of all the subsystems the system failure probability is

$$F_{AFTER} = \frac{1}{2}F_1 + \frac{1}{8}F_2F_3F_4 - \frac{1}{16}F_1F_2F_3F_4$$

The effect on system reliability may be gauged by the difference

$$R_{AFTER} - R_{BEFORE} = \frac{1}{2}F_1 + \frac{7}{8}F_2F_3F_4 - \frac{15}{16}F_1F_2F_3F_4$$

Or by the ratio

$$\frac{R_{AFTER}}{R_{BEFORE}} = \frac{1 - \left[\frac{1}{2}F_1 + \frac{1}{8}F_2F_3F_4 - \frac{1}{16}F_1F_2F_3F_4 \right]}{1 - \left[F_1 + F_2F_3F_4 - F_1F_2F_3F_4 \right]}$$

These formula do not provide much insight into the behavior of the system. It is useful to consider two possible simplifications:

1) If the failure probabilities are all of comparable magnitudes and are $\ll 1$, one may safely ignore the higher order terms yielding.

$$R_{BEFORE} \approx 1 - F_1 \qquad R_{AFTER} \approx 1 - \frac{1}{2} F_1$$

$$R_{AFTER} - R_{BEFORE} \approx \frac{1}{2} F_1 \qquad \frac{R_{AFTER}}{R_{BEFORE}} \approx 1 + \frac{1}{2} F_1$$

So, the reliability after improvement is twice as close to 1 as it was before improvement.

2) It may be more reasonable to assume that F_1 and $F_2F_3F_4$ are comparable but small. After all, why would someone put all those systems in parallel unless they were the weak link in the system?

Under these conditions

$$R_{BEFORE} \approx 1 - F_1 - F_2F_3F_4 \qquad R_{AFTER} \approx 1 - \frac{1}{2} F_1 - \frac{1}{8} F_2F_3F_4$$

$$R_{AFTER} - R_{BEFORE} \approx \frac{1}{2} F_1 + \frac{7}{8} F_2F_3F_4 \qquad \frac{R_{AFTER}}{R_{BEFORE}} \approx 1 + \frac{1}{2} F_1 + \frac{7}{8} F_2F_3F_4$$