

Temperature Controller Simulation

Coded by Dan Frey, 2 July 1998

This Mathcad sheet reproduces the results from Padke, "Quality Engineering Using Robust Design" chapter 9. The sheet is intended to facilitate deeper exploration of the case study and to help 16.881 students complete homework #8 -- "Design of Dynamic Systems".

ORIGIN := 1 The first index of all vectors and matrices will be one.

Define the governing equations of the temperature controller circuit. An explanation of the circuit can be found on pg. 214-216 of Phadke. The governing equations are on page 218.

Define the System Model

$E_0 := 10$ The supply voltage is not a control factor, so its nominal value is set right up front. There will be noise induced about the nominal however.

This is the governing equation for RT-ON found in Phadke.

$$RT_ON(R_1, R_2, R_3, R_4, E_0, E_Z) := \frac{R_3 \cdot R_2 \cdot (E_Z \cdot R_4 + E_0 \cdot R_1)}{R_1 \cdot (E_Z \cdot R_2 + E_Z \cdot R_4 - E_0 \cdot R_2)}$$

This second equation defines the value of RT-ON as a function of vectors of signal factors (SF), noise factors (NF), and control factors (CF).

$$RT_ON2(SF, NF, CF) := RT_ON(NF_1 \cdot CF_1, NF_2 \cdot CF_2, SF, NF_3 \cdot CF_3, NF_4 \cdot E_0, NF_5 \cdot CF_4)$$

This is the governing equation for RT-OFF found in Phadke.

$$RT_OFF(R_1, R_2, R_3, R_4, E_0, E_Z) := \frac{R_3 \cdot R_2 \cdot R_4}{R_1 \cdot (R_2 + R_4)}$$

This second equation defines the value of RT-OFF as a function of vectors of signal factors (SF), noise factors (NF), and control factors (CF).

$$RT_OFF2(SF, NF, CF) := RT_OFF(NF_1 \cdot CF_1, NF_2 \cdot CF_2, SF, NF_3 \cdot CF_3, NF_4 \cdot E_0, NF_5 \cdot CF_4)$$

Choose Noise, Control, and Signal Factor Levels

These are the noise levels defined in Table 9.1 on page 220. I've switched the ordering so that the column numbers correspond to the compound noise factor levels defined on page 221.

$$\text{noise_levels} := \begin{pmatrix} 1.0204 & 1 & 0.9796 \\ 0.9796 & 1 & 1.0204 \\ 1.0204 & 1 & 0.9796 \\ 0.9796 & 1 & 1.0204 \\ 1.0204 & 1 & 0.9796 \end{pmatrix}$$

Here are the signal factor levels as defined in Table 9.2.

$$\text{signal_levels} := \begin{pmatrix} 0.5 \\ 1 \\ 1.5 \end{pmatrix}$$

Here are the control factor level assignments within the L18 as defined by Tables 9.3 and 9.4.

$$\text{cf_level} := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2.668 & 4.0 & 6.0 \\ 5.336 & 8.0 & 12.0 \\ 26.68 & 40 & 60 \\ 1 & 2 & 3 \\ 4.8 & 6 & 7.2 \\ 1 & 2 & 3 \end{pmatrix}$$

This command forms the L18 matrix.

$$L_{18} := \text{stack} \left[\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 2 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 2 & 3 & 3 & 1 & 1 \\ 1 & 2 & 3 & 3 & 1 & 1 & 2 & 2 \end{pmatrix}, \text{stack} \left[\begin{pmatrix} 1 & 3 & 1 & 2 & 1 & 3 & 2 & 3 \\ 1 & 3 & 2 & 3 & 2 & 1 & 3 & 1 \\ 1 & 3 & 3 & 1 & 3 & 2 & 1 & 2 \\ 2 & 1 & 1 & 3 & 3 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 & 1 & 3 & 3 & 2 \\ 2 & 1 & 3 & 2 & 2 & 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 1 & 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 2 & 3 & 1 & 2 & 1 & 3 \\ 2 & 2 & 3 & 1 & 2 & 3 & 2 & 1 \\ 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 \\ 2 & 3 & 2 & 1 & 3 & 1 & 2 & 3 \\ 2 & 3 & 3 & 2 & 1 & 2 & 3 & 1 \end{pmatrix} \right]$$

Perform the Experiment

For every row of the experiment, compute the values of RT-ON. The values of RT_ON are made by running a full factorial on the signal factor and the compound noise factor. This is easily accomplished with nested FOR loops. Within the FOR loops, single values of RT-ON are computed using the L18 matrix and control factor level table.

row := 1..18

$$\text{RON}_{\text{row}} := \left[\begin{array}{l} \text{for signal} \in 1..3 \\ \text{for noise} \in 1..3 \\ \\ y_{\text{signal, noise}} \leftarrow \text{RT_ON2} \left[\text{signal_levels}_{\text{signal}}, \text{noise_levels}_{\langle \text{noise} \rangle}, \left(\begin{array}{l} \text{cf_level}_{3, L_{18, \text{row}, 3}} \\ \text{cf_level}_{4, L_{18, \text{row}, 4}} \\ \text{cf_level}_{5, L_{18, \text{row}, 5}} \\ \text{cf_level}_{7, L_{18, \text{row}, 7}} \end{array} \right) \right] \\ \\ y \end{array} \right]$$

$$\text{RON}_1 = \begin{pmatrix} 1.426 & 1.543 & 1.676 \\ 2.852 & 3.085 & 3.353 \\ 4.277 & 4.628 & 5.029 \end{pmatrix}$$

Here is an example of the data from a single row (the first row) of the inner array. Rows correspond to signal factor levels and noises correspond to columns. There is greater variation due to the signal than the noise which is as it should be.

Compute the values of beta, error variance, and S/N ratio directly from the equations for the Continuous-continuous case as given on Phadke page 115.

$$\beta_{\text{ON}_{\text{row}}} := \frac{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 \left(R_{\text{ON}_{\text{row}}} \right)_{\text{signal, noise}} \cdot \text{signal_levels}_{\text{signal}}}{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 \left(\text{signal_levels}_{\text{signal}} \right)^2}$$

$$\sigma_{\text{eON}_{\text{row}}}^2 := \frac{1}{3 \cdot 3 - 1} \cdot \sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 \left[\left(R_{\text{ON}_{\text{row}}} \right)_{\text{signal, noise}} - \beta_{\text{ON}_{\text{row}}} \cdot \text{signal_levels}_{\text{signal}} \right]^2$$

$$\eta_{\text{ON}_{\text{row}}} := 10 \cdot \log \left[\frac{\left(\beta_{\text{ON}_{\text{row}}} \right)^2}{\sigma_{\text{eON}_{\text{row}}}^2} \right]$$

Repeat the exact same process for the second response RT-OFF.

For every row of the experiment, compute the values of RT-OFF. The values of RT_OFF are made by running a full factorial on the signal factor and the compound noise factor. This is easily accomplished with nested FOR loops. Within the FOR loops, single values of RT-OFF are computed using the L18 matrix and control factor level table.

```

ROFF_row :=
  for signal ∈ 1..3
  for noise ∈ 1..3
    y_signal, noise ← RT_OFF2 [ signal_levels_signal, noise_levels_noise,
      ( cf_level_3, L18_row,3
        cf_level_4, L18_row,4
        cf_level_5, L18_row,5
        cf_level_7, L18_row,7 ) ]
  y

```

$$ROFF_1 = \begin{pmatrix} 0.805 & 0.833 & 0.862 \\ 1.611 & 1.667 & 1.724 \\ 2.416 & 2.5 & 2.586 \end{pmatrix}$$

Here is an example of the data from a single row (the first row) of the inner array. Rows correspond to signal row levels and noises correspond to columns. There is greater variation due to the signal than the noise which is as it should be.

Compute the values of beta, error variance, and S/N ratio directly from the equations for the Continuous-continuous case as given on Phadke page 115.

$$\beta_{\text{OFF}_{\text{row}}} := \frac{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 \left(R_{\text{OFF}_{\text{row}}} \right)_{\text{signal, noise}} \cdot \text{signal_levels}_{\text{signal}}}{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 \left(\text{signal_levels}_{\text{signal}} \right)^2}$$

$$\sigma_{\text{eOFF}_{\text{row}}}^2 := \frac{1}{3 \cdot 3 - 1} \cdot \sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 \left[\left(R_{\text{OFF}_{\text{row}}} \right)_{\text{signal, noise}} - \beta_{\text{OFF}_{\text{row}}} \cdot \text{signal_levels}_{\text{signal}} \right]^2$$

$$\eta_{\text{OFF}_{\text{row}}} := 10 \cdot \log \left[\frac{\left(\beta_{\text{OFF}_{\text{row}}} \right)^2}{\sigma_{\text{eOFF}_{\text{row}}}^2} \right]$$

Define a function that will perform ANOVA for any response at any factor at all levels. The first argument is the name of the data set to be analyzed. The second argument is the number of the control factor. This will be helpful in creating the factor effect plots below.

```

m(response, factor_no) :=
  for level ∈ 1..3
  |
  | sum ← 0
  | for row ∈ 1..18
  |   sum ← sum + response_row if L18_row, factor_no = level
  | mean_level ←  $\frac{\text{sum}}{6}$ 
  |
  | mean

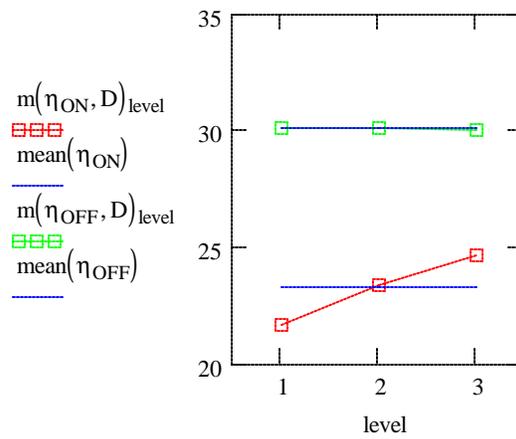
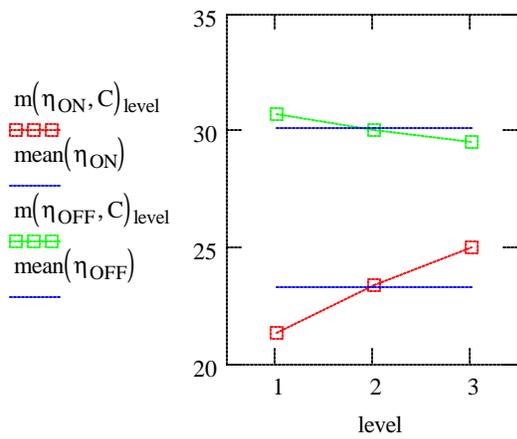
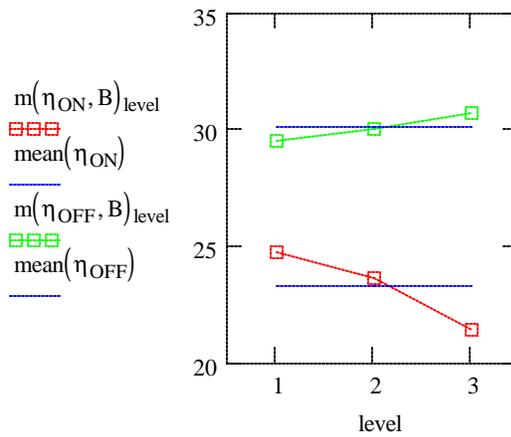
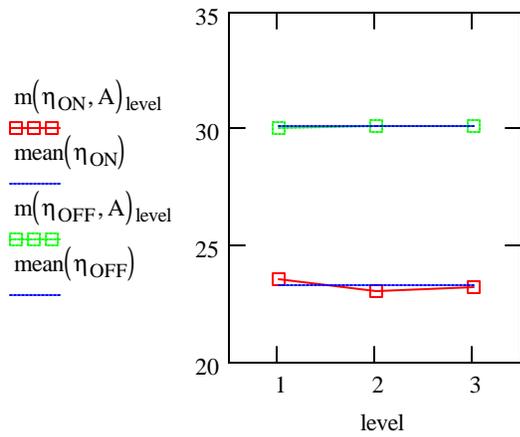
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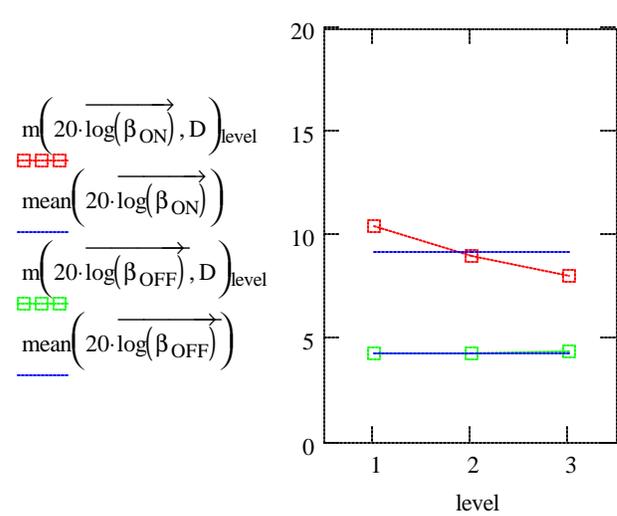
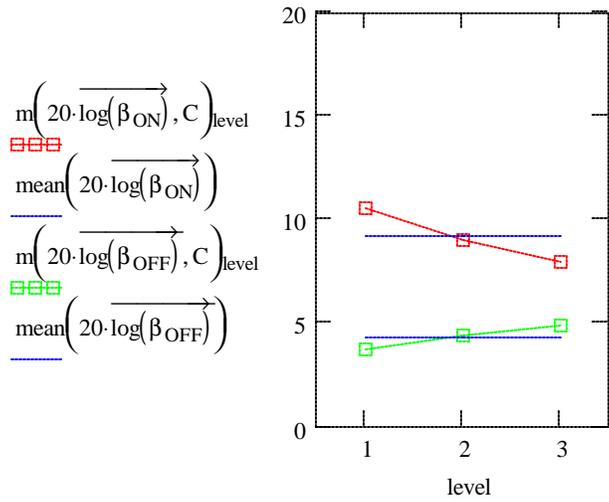
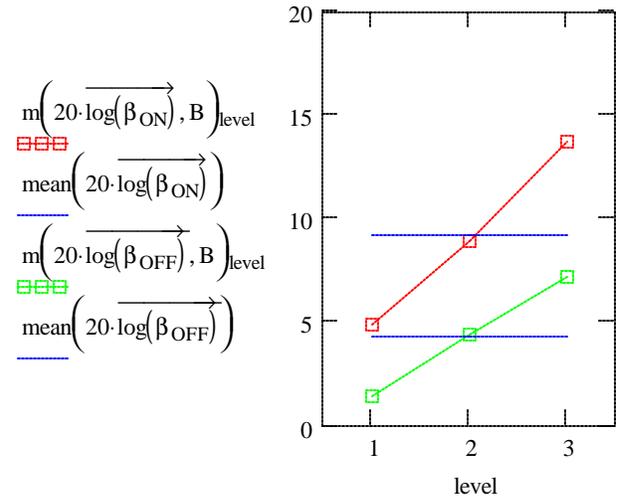
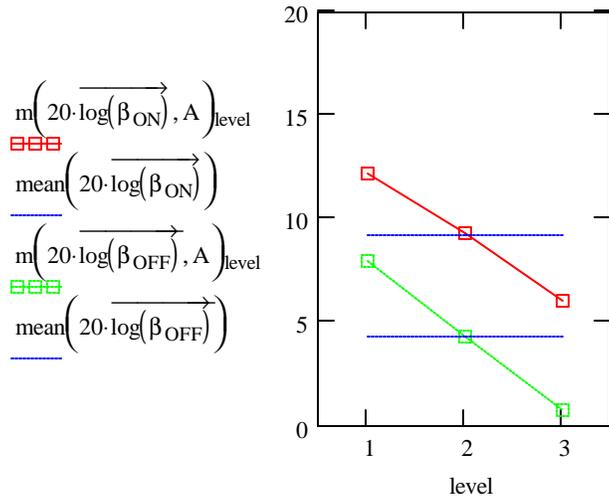
Define the range variable to step through the three level factors.

level := 1..3

Create a shorthand notation for the control factors consistent with Table 9.4.

A := 3 B := 4 C := 5 D := 7





Optimization

The optimum settings chosen by Padke were:

A=2 B=1 C=3 D=3

For RT-ON he prediction is:

$$\eta_{\text{pred}} := \text{mean}(\eta_{\text{ON}}) + m(\eta_{\text{ON}}, A)_2 + m(\eta_{\text{ON}}, B)_1 + m(\eta_{\text{ON}}, C)_3 + m(\eta_{\text{ON}}, D)_3 - 4 \cdot \text{mean}(\eta_{\text{ON}})$$

$$\eta_{\text{pred}} = 27.701$$

Confirmation

The true value is:

Run through all the signal and noise factors at the optimal control factor settings.

$$\text{RON_OPT} := \left[\begin{array}{l} \text{for signal} \in 1..3 \\ \text{for noise} \in 1..3 \\ y_{\text{signal, noise}} \leftarrow \text{RT_ON2} \left[\text{signal_levels}_{\text{signal}}, \text{noise_levels}_{\langle \text{noise} \rangle}, \begin{pmatrix} 4.0 \\ 5.336 \\ 60.0 \\ 7.2 \end{pmatrix} \right] \\ y \end{array} \right.$$

$$\text{RON_OPT} = \begin{pmatrix} 0.718 & 0.755 & 0.795 \\ 1.435 & 1.51 & 1.589 \\ 2.153 & 2.265 & 2.384 \end{pmatrix} \quad \text{Here is the data you'd gather.}$$

$$\beta_{\text{ON_OPT}} := \frac{\sum_{\text{signal}=1}^3 \sum_{\text{noise}=1}^3 (\text{RON_OPT})_{\text{signal, noise}} \cdot \text{signal_levels}_{\text{signal}}}{\sum_{\text{signal}=1}^3 \sum_{\text{noise}=1}^3 (\text{signal_levels}_{\text{signal}})^2}$$

$$\sigma_{e\text{ON_OPT}}^2 := \frac{1}{3 \cdot 3 - 1} \cdot \sum_{\text{signal}=1}^3 \sum_{\text{noise}=1}^3 \left[(\text{RON_OPT})_{\text{signal, noise}} - \beta_{\text{ON_OPT}} \cdot \text{signal_levels}_{\text{signal}} \right]^2$$

$$\eta_{\text{ON_OPT}} := 10 \cdot \log \left[\frac{(\beta_{\text{ON_OPT}})^2}{\sigma_{e\text{ON_OPT}}^2} \right]$$

$$\eta_{\text{ON_OPT}} = 26.436$$

$$\eta_{\text{pred}} = 27.701$$

$$\text{mean}(\eta_{\text{ON}}) = 23.241$$

Not a bad prediction. The forecast was for an increase of about 4.4 dB and we actually realized an increase of 3.2 dB.

Solution

a) At the optimized control factor settings, what is the variance in RT-ON at the three settings of the signal factor R3?

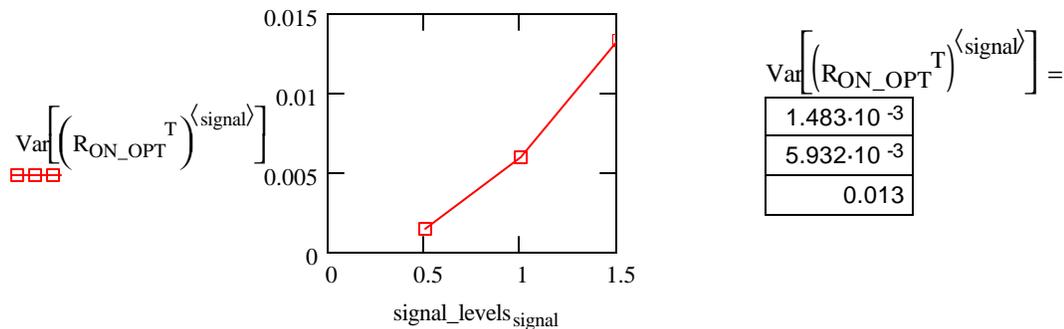
Run through all the signal and noise factors at the optimal control factor settings.

$$\text{RON_OPT} := \begin{cases} \text{for signal} \in 1..3 \\ \text{for noise} \in 1..3 \\ y_{\text{signal, noise}} \leftarrow \text{RT_ON2} \left[\text{signal_levels}_{\text{signal}}, \text{noise_levels}_{\langle \text{noise} \rangle}, \begin{pmatrix} 4.0 \\ 5.336 \\ 60.0 \\ 7.2 \end{pmatrix} \right] \\ y \end{cases}$$

$$\text{RON_OPT} = \begin{pmatrix} 0.718 & 0.755 & 0.795 \\ 1.435 & 1.51 & 1.589 \\ 2.153 & 2.265 & 2.384 \end{pmatrix} \quad \text{Here is the data you'd gather.}$$

signal := 1..3

Plot and tabulate the variance at the various signal factor levels.



There is a roughly linear scaling of variance with the signal factor R3. This confirms what we stated in lecture 12, that robust design generally assumes this will hold making S/N a useful metric (independent of adjustment with the scaling factor).

b) Run a Monte Carlo Simulation to check the results from part (a).

number_of_trials := 1000

$R_1 := \text{rnorm}\left(\text{number_of_trials}, 4.0, 4.0 \cdot \frac{5\%}{3}\right)$

$R_2 := \text{rnorm}\left(\text{number_of_trials}, 5.336, 5.336 \cdot \frac{5\%}{3}\right)$

$R_3 := \text{signal_levels}$

$R_4 := \text{rnorm}\left(\text{number_of_trials}, 60.0, 60.0 \cdot \frac{5\%}{3}\right)$

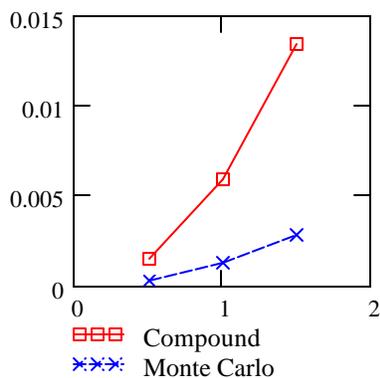
$E_z := \text{rnorm}\left(\text{number_of_trials}, 7.2, 7.2 \cdot \frac{5\%}{3}\right)$

$E_o := \text{rnorm}\left(\text{number_of_trials}, 10, 10 \cdot \frac{5\%}{3}\right)$

trial := 1.. number_of_trials

$R_{ON_MC_{\text{trial, signal}}} := R_{T_ON}(R_{1_{\text{trial}}}, R_{2_{\text{trial}}}, R_{3_{\text{signal}}}, R_{4_{\text{trial}}}, E_{o_{\text{trial}}}, E_{z_{\text{trial}}})$

$VAR_{MC_{\text{signal}}} := \text{Var}(R_{ON_MC}^{\langle \text{signal} \rangle})$



The compounding strategy is just about 5 times too overconservative at every signal factor level. The Monte Carlo simulation takes more function evaluations, but the variance values are much more accurate and coding is quite a bit simpler. Bottom line lesson, compounding noise factors is a good strategy in parameter design because it drives you to the right control factor settings, but is too overconservative for tolerance design.

c) Use a different noise strategy. Pick an appropriate outer array and induce the noise through it.

There are five noise factors at three levels. The L18 or the L16 primed would both be fine. I will use the L18 since I've already entered the definition of the matrix.

Perform the Experiment

For every row of the experiment, compute the values of RT-ON. The values of RT_ON are made by running a full factorial on the signal factor and the compound noise factor. This is easily accomplished with nested FOR loops. Within the FOR loops, single values of RT-ON are computed using the L18 matrix and control factor level table.

row := 1.. 18

$$\text{noise_levels} := \begin{pmatrix} 0.98 & 1 & 1.02 \\ 0.98 & 1 & 1.02 \\ 0.98 & 1 & 1.02 \\ 0.98 & 1 & 1.02 \\ 0.98 & 1 & 1.02 \end{pmatrix}$$

$$\text{RON18}_{\text{row}} := \begin{array}{l} \text{for signal} \in 1..3 \\ \quad \text{for noise} \in 1..18 \\ \quad \quad y_{\text{signal, noise}} \leftarrow \text{RT_ON2} \left[\text{signal_levels}_{\text{signal}}, \begin{pmatrix} \text{noise_levels}_{1, L_{18} \text{ noise}, 2} \\ \text{noise_levels}_{2, L_{18} \text{ noise}, 3} \\ \text{noise_levels}_{3, L_{18} \text{ noise}, 4} \\ \text{noise_levels}_{4, L_{18} \text{ noise}, 5} \\ \text{noise_levels}_{5, L_{18} \text{ noise}, 6} \end{pmatrix}, \begin{pmatrix} \text{cf_level}_{3, L_{18} \text{ row}, 3} \\ \text{cf_level}_{4, L_{18} \text{ row}, 4} \\ \text{cf_level}_{5, L_{18} \text{ row}, 5} \\ \text{cf_level}_{7, L_{18} \text{ row}, 7} \end{pmatrix} \right] \end{array}$$

Here is an example of the data from a single row (the first row) of the inner array. Rows correspond to signal factor levels and noises correspond to columns. Note that now there are 18 columns of noise.

$$R_{ON18_1} =$$

	1	2	3	4	5	6	7	8	9	10
1	1.543	1.569	1.595	1.517	1.543	1.568	1.44	1.526	1.595	1.536
2	3.085	3.137	3.189	3.034	3.085	3.136	2.879	3.051	3.189	3.073
3	4.628	4.706	4.784	4.551	4.628	4.704	4.319	4.577	4.784	4.609

Compute the values of beta, error variance, and S/N ratio directly from the equations for the Continuous-continuous case as given on Phadke page 115.

row := 1.. 18

$$\beta_{ON18_{row}} := \frac{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^{18} (R_{ON18_{row}})_{\text{signal, noise}} \cdot \text{signal_levels}_{\text{signal}}}{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^{18} (\text{signal_levels}_{\text{signal}})^2}$$

$$\sigma_{eON18_{row}}^2 := \frac{1}{3 \cdot 18 - 1} \cdot \sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^{18} \left[(R_{ON18_{row}})_{\text{signal, noise}} - \beta_{ON18_{row}} \cdot \text{signal_levels}_{\text{signal}} \right]^2$$

$$\eta_{ON18_{row}} := 10 \cdot \log \left[\frac{(\beta_{ON18_{row}})^2}{\sigma_{eON18_{row}}^2} \right]$$

Repeat the exact same process for the second response RT-OFF.

For every row of the experiment, compute the values of RT-OFF. The values of RT_OFF are made by running a full factorial on the signal factor and the compound noise factor. This is easily accomplished with nested FOR loops. Within the FOR loops, single values of RT-OFF are computed using the L18 matrix and control factor level table.

$$\text{ROFF18}_{\text{row}} := \begin{array}{l} \text{for signal} \in 1..3 \\ \quad \text{for noise} \in 1..18 \\ \quad \quad y_{\text{signal, noise}} \leftarrow \text{RT_OFF2}(\text{signal_levels}_{\text{signal}}, \left(\begin{array}{l} \text{noise_levels}_{1, L_{18} \text{ noise}, 2} \\ \text{noise_levels}_{2, L_{18} \text{ noise}, 3} \\ \text{noise_levels}_{3, L_{18} \text{ noise}, 4} \\ \text{noise_levels}_{4, L_{18} \text{ noise}, 5} \\ \text{noise_levels}_{5, L_{18} \text{ noise}, 6} \end{array} \right), \left(\begin{array}{l} \text{cf_level}_{3, L_{18} \text{ row}, 3} \\ \text{cf_level}_{4, L_{18} \text{ row}, 4} \\ \text{cf_level}_{5, L_{18} \text{ row}, 5} \\ \text{cf_level}_{7, L_{18} \text{ row}, 7} \end{array} \right) \\ \quad \quad y \end{array}$$

Here is an example of the data from a single row (the first row) of the inner array. Rows correspond to signal factor levels and noises correspond to columns. Note that now there are 18 columns of noise.

	1	2	3	4	5	6	7	8	9	10
ROFF18 ₁	0.833	0.85	0.867	0.817	0.833	0.85	0.803	0.82	0.828	0.839
	1.667	1.701	1.735	1.633	1.667	1.7	1.607	1.639	1.655	1.678
	2.5	2.551	2.602	2.45	2.5	2.55	2.41	2.459	2.483	2.516

Compute the values of beta, error variance, and S/N ratio directly from the equations for the Continuous-continuous case as given on Phadke page 115.

$$\beta_{\text{OFF18}_{\text{row}}} := \frac{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^{18} \left(R_{\text{OFF18}_{\text{row}}} \right)_{\text{signal, noise}} \cdot \text{signal_levels}_{\text{signal}}}{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^{18} \left(\text{signal_levels}_{\text{signal}} \right)^2}$$

$$\sigma_{\text{eOFF18}_{\text{row}}}^2 := \frac{1}{3 \cdot 18 - 1} \cdot \sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^{18} \left[\left(R_{\text{OFF18}_{\text{row}}} \right)_{\text{signal, noise}} - \beta_{\text{OFF18}_{\text{row}}} \cdot \text{signal_levels}_{\text{signal}} \right]^2$$

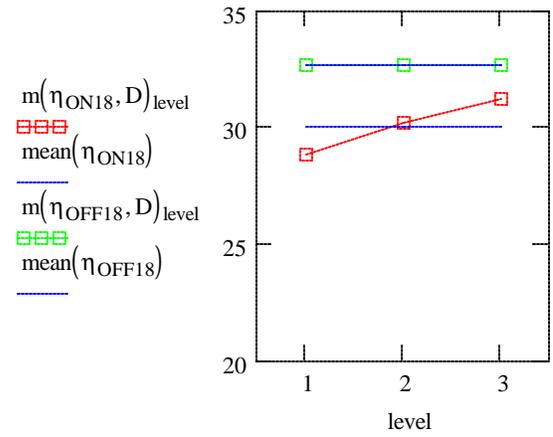
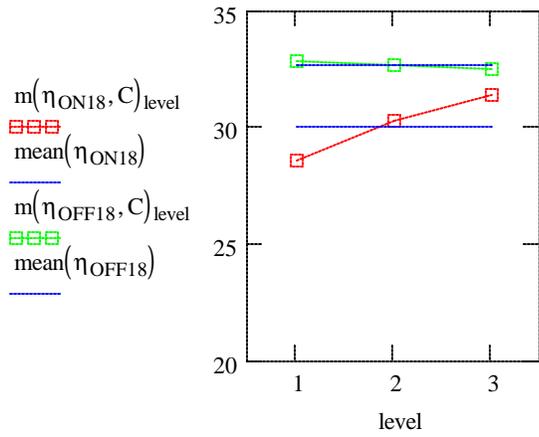
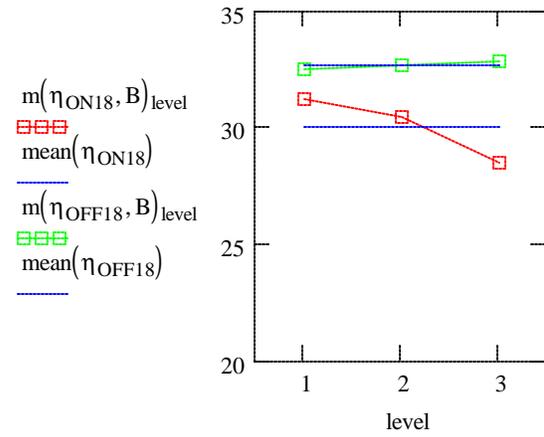
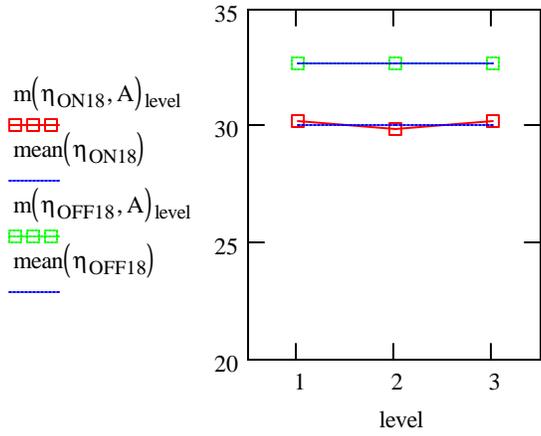
$$\eta_{\text{OFF18}_{\text{row}}} := 10 \cdot \log \left[\frac{\left(\beta_{\text{OFF18}_{\text{row}}} \right)^2}{\sigma_{\text{eOFF18}_{\text{row}}}^2} \right]$$

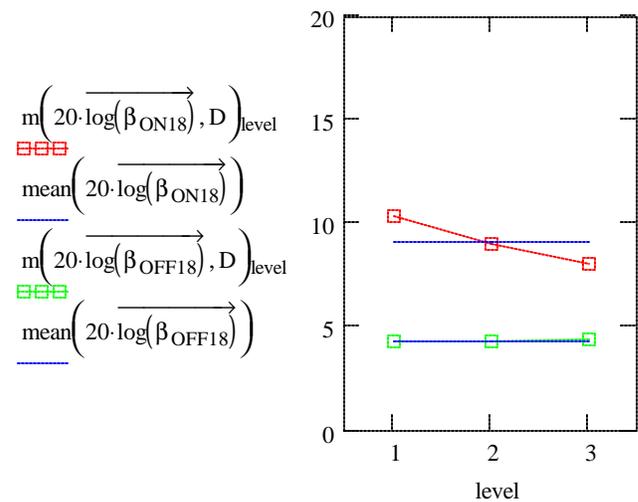
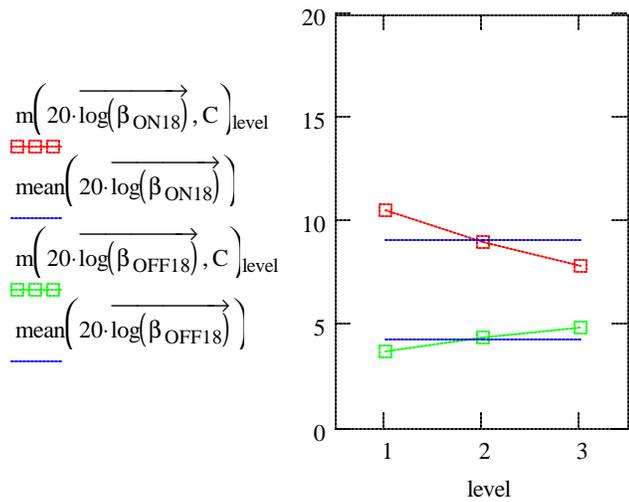
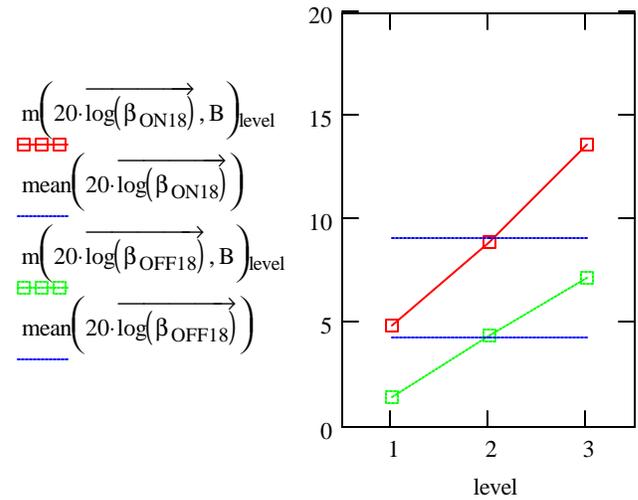
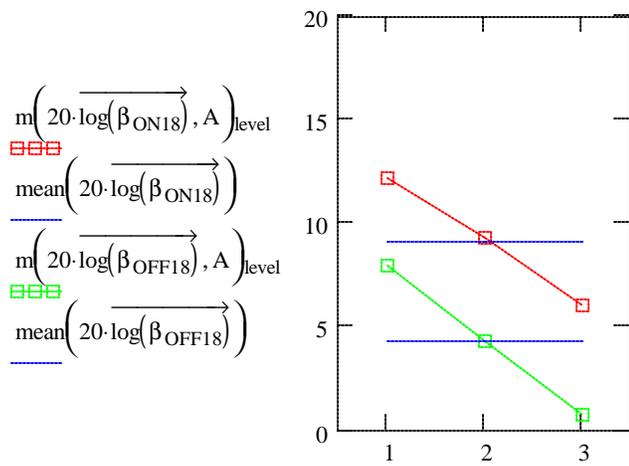
Define the range variable to step through the three level factors.

level := 1.. 3

Create a shorthand notation for the control factors consistent with Table 9.4.

A := 3 B := 4 C := 5 D := 7



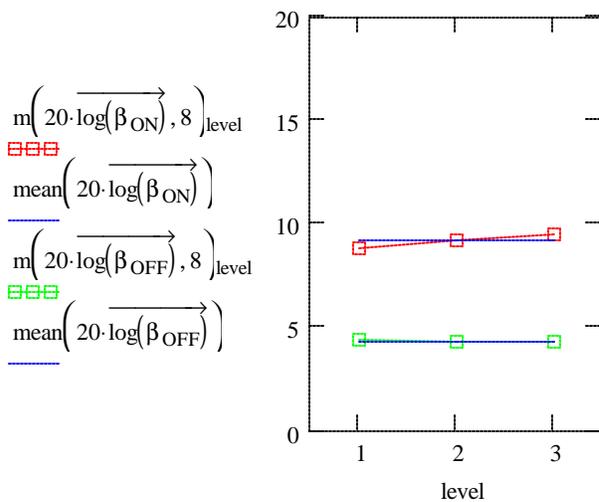
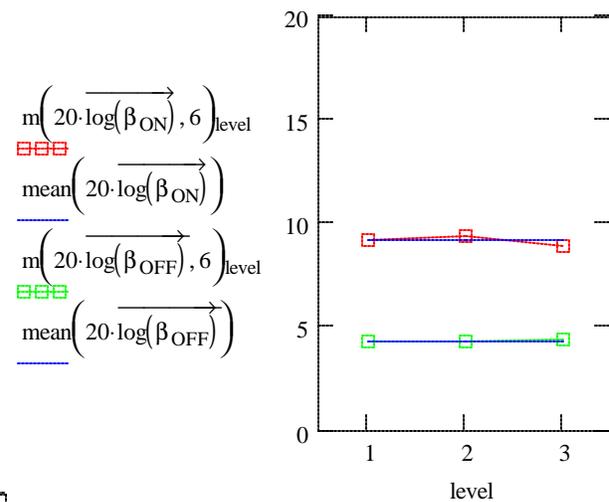
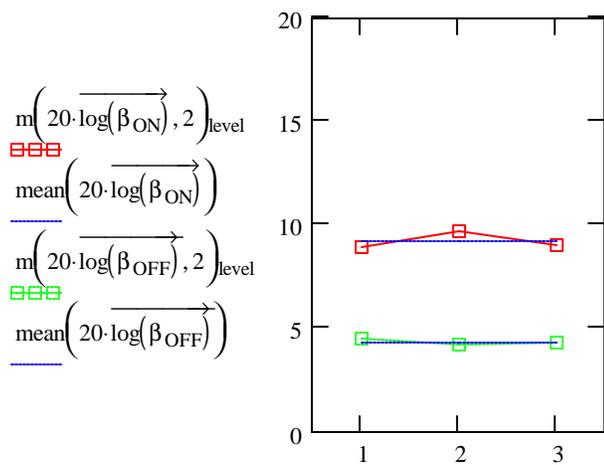


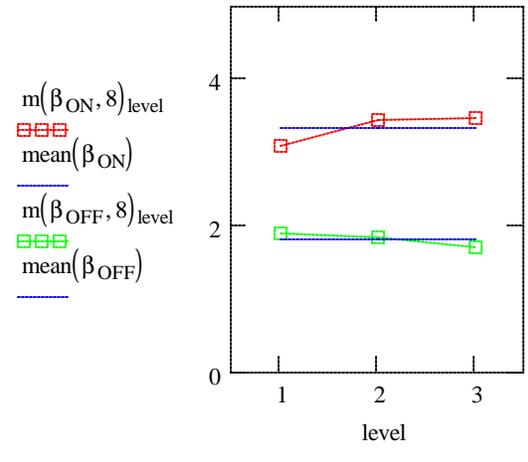
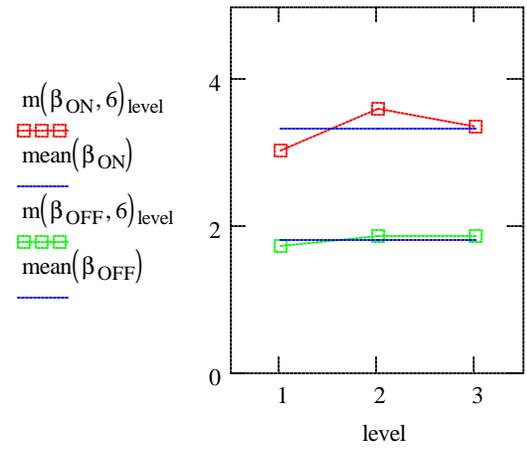
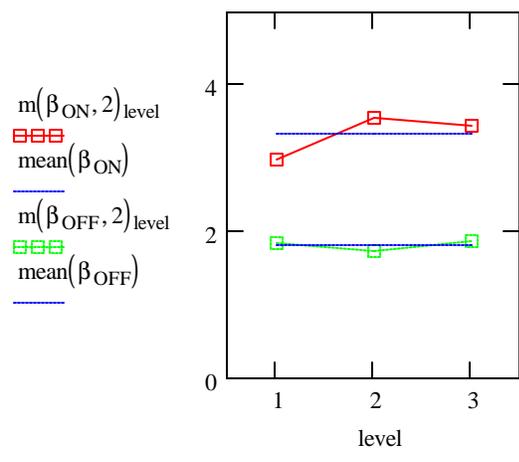
Note that the S/N ratios all went up (about 5dB), but the shapes of the curves were preserved. This means that the decisions you would make regarding control factor settings would be unaffected by the change in experimental procedure.

Also note that the plots of $20\log(\beta)$ were unaffected by the change in noise strategy. The overconservative nature of noise compounding does not seem to bias the results wrt to β . The same trend probably holds for the ANOM on the mean for static experiments.

d) Phadke states that the additive model holds better for $20\log(\beta)$ than for β . Provide an analysis to show that this is the case.

one way to check for additivity is to look at the variance within the unassigned columns. These columns are estimates of error variance AND interactions. All other things being equal, if a response is more additive, then the unassigned columns will be more nearly flat.





Another way to solve this problem is to consider the difference between the prediction and the confirmation.

The true value is:

Run through all the signal and noise factors at the optimal control factor settings.

$$\text{RON_OPT} := \begin{cases} \text{for signal} \in 1..3 \\ \text{for noise} \in 1..3 \\ y_{\text{signal, noise}} \leftarrow \text{RT_ON2} \left[\text{signal_levels}_{\text{signal}}, \text{noise_levels}_{\langle \text{noise} \rangle}, \begin{pmatrix} 4.0 \\ 5.336 \\ 60.0 \\ 7.2 \end{pmatrix} \right] \\ y \end{cases}$$

$$\text{RON_OPT} = \begin{pmatrix} 0.755 & 0.755 & 0.755 \\ 1.51 & 1.51 & 1.51 \\ 2.265 & 2.265 & 2.265 \end{pmatrix} \quad \text{Here is the data you'd gather.}$$

$$\beta_{\text{ON_OPT}} := \frac{\sum_{\text{signal}=1}^3 \sum_{\text{noise}=1}^3 (\text{RON_OPT})_{\text{signal, noise}} \cdot \text{signal_levels}_{\text{signal}}}{\sum_{\text{signal}=1}^3 \sum_{\text{noise}=1}^3 (\text{signal_levels}_{\text{signal}})^2}$$

$$\beta_{\text{ON_OPT}} = 1.51$$

$$20 \cdot \log(\beta_{\text{ON_OPT}}) = 3.578$$

For beta, the prediction is:

$$\beta_{\text{pred}} := -3 \cdot \text{mean}(\beta_{\text{ON}}) + m(\beta_{\text{ON}}, A)_2 + m(\beta_{\text{ON}}, B)_1 + m(\beta_{\text{ON}}, C)_3 + m(\beta_{\text{ON}}, D)_3$$

$$\beta_{\text{pred}} = 0.719$$

$$\frac{\beta_{\text{ON_OPT}} - \beta_{\text{pred}}}{\beta_{\text{ON_OPT}}} = 52.388 \%$$

For the $20\log(\beta)$ metric, the prediction is:

$$\log\beta_{\text{pred}} := -3 \cdot \text{mean}\left(\overrightarrow{20 \cdot \log(\beta_{\text{ON}})}\right) + m\left(\overrightarrow{20 \cdot \log(\beta_{\text{ON}})}, A\right)_2 + m\left(\overrightarrow{20 \cdot \log(\beta_{\text{ON}})}, B\right)_1 + m\left(\overrightarrow{20 \cdot \log(\beta_{\text{ON}})}, C\right)_3 + m\left(\overrightarrow{20 \cdot \log(\beta_{\text{ON}})}, D\right)_4$$

$$\log\beta_{\text{pred}} = 2.612$$

$$\frac{20 \cdot \log(\beta_{\text{ON_OPT}}) - \log\beta_{\text{pred}}}{20 \cdot \log(\beta_{\text{ON_OPT}})} = 27.003 \%$$

So, in fact, the log transform on beta improved additivity in this case.

e) Carry out one additional iteration of robustness and discuss your results.

Here are the previous control factor level assignments within the L18 as defined by Tables 9.3 and 9.4.

$$\text{cf_level} := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2.668 & 4.0 & 6.0 \\ 5.336 & 8.0 & 12.0 \\ 26.68 & 40 & 60 \\ 1 & 2 & 3 \\ 4.8 & 6 & 7.2 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{noise_levels} := \begin{pmatrix} 1.0204 & 1 & 0.9796 \\ 0.9796 & 1 & 1.0204 \\ 1.0204 & 1 & 0.9796 \\ 0.9796 & 1 & 1.0204 \\ 1.0204 & 1 & 0.9796 \end{pmatrix}$$

Phadke states that in iterative optimization, you should take the optimum value as the baseline, middle value. Then, he says, set the high and low values according to the same relationships that were previously assigned. Referring to section 9.4 (pg. 222), you must set the levels for the second iteration as:

$$\text{cf_level} := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ \frac{4.0}{1.5} & 4.0 & 1.5 \cdot 4.0 \\ \frac{5.336}{1.5} & 5.336 & 1.5 \cdot 5.336 \\ \frac{60}{1.5} & 60 & 1.5 \cdot 60 \\ 1 & 2 & 3 \\ 0.8 \cdot 7.2 & 7.2 & 7.2 \cdot 1.2 \\ 1 & 2 & 3 \end{pmatrix}$$

Perform the Experiment

For every row of the experiment, compute the values of RT-ON. The values of RT_ON are made by running a full factorial on the signal factor and the compound noise factor. This is easily accomplished with nested FOR loops. Within the FOR loops, single values of RT-ON are computed using the L18 matrix and control factor level table.

row := 1..18

$$\text{RON}_{\text{row}} := \left[\begin{array}{l} \text{for signal} \in 1..3 \\ \quad \text{for noise} \in 1..3 \\ \\ y_{\text{signal,noise}} \leftarrow \text{RT_ON2} \left(\text{signal_levels}_{\text{signal}}, \text{noise_levels}_{\langle \text{noise} \rangle}, \begin{pmatrix} \text{cf_level}_{3, L_{18, \text{row}, 3}} \\ \text{cf_level}_{4, L_{18, \text{row}, 4}} \\ \text{cf_level}_{5, L_{18, \text{row}, 5}} \\ \text{cf_level}_{7, L_{18, \text{row}, 7}} \end{pmatrix} \right) \end{array} \right]$$

$$\text{RON}_1 = \begin{pmatrix} 0.754 & 0.796 & 0.842 \\ 1.509 & 1.593 & 1.683 \\ 2.263 & 2.389 & 2.525 \end{pmatrix}$$

Here is an example of the data from a single row (the first row) of the inner array. Rows correspond to signal factor levels and noises correspond to columns. There is greater variation due to the signal than the noise which is as it should be.

Compute the values of beta, error variance, and S/N ratio directly from the equations for the Continuous-continuous case as given on Phadke page 115.

$$\beta_{\text{ON}_{\text{row}}} := \frac{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 \left(R_{\text{ON}_{\text{row}}} \right)_{\text{signal, noise}} \cdot \text{signal_levels}_{\text{signal}}}{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 \left(\text{signal_levels}_{\text{signal}} \right)^2}$$

$$\sigma_{\text{eON}_{\text{row}}}^2 := \frac{1}{3 \cdot 3 - 1} \cdot \sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 \left[\left(R_{\text{ON}_{\text{row}}} \right)_{\text{signal, noise}} - \beta_{\text{ON}_{\text{row}}} \cdot \text{signal_levels}_{\text{signal}} \right]^2$$

$$\eta_{\text{ON}_{\text{row}}} := 10 \cdot \log \left[\frac{\left(\beta_{\text{ON}_{\text{row}}} \right)^2}{\sigma_{\text{eON}_{\text{row}}}^2} \right]$$

Repeat the exact same process for the second response RT-OFF.

For every row of the experiment, compute the values of RT-OFF. The values of RT_OFF are made by running a full factorial on the signal factor and the compound noise factor. This is easily accomplished with nested FOR loops. Within the FOR loops, single values of RT-OFF are computed using the L18 matrix and control factor level table.

$$\text{ROFF}_{\text{row}} := \left[\begin{array}{l} \text{for signal} \in 1..3 \\ \quad \text{for noise} \in 1..3 \\ \\ y_{\text{signal, noise}} \leftarrow \text{RT_OFF2} \left[\text{signal_levels}_{\text{signal}}, \text{noise_levels}_{\langle \text{noise} \rangle}, \left(\begin{array}{l} \text{cf_level}_{3, L_{18, \text{row}, 3}} \\ \text{cf_level}_{4, L_{18, \text{row}, 4}} \\ \text{cf_level}_{5, L_{18, \text{row}, 5}} \\ \text{cf_level}_{7, L_{18, \text{row}, 7}} \end{array} \right) \right] \\ \\ y \end{array} \right.$$

$$\text{ROFF}_1 = \begin{pmatrix} 0.59 & 0.613 & 0.636 \\ 1.18 & 1.225 & 1.272 \\ 1.77 & 1.838 & 1.908 \end{pmatrix}$$

Here is an example of the data from a single row (the first row) of the inner array. Rows correspond to signal row levels and noises correspond to columns. There is greater variation due to the signal than the noise which is as it should be.

Compute the values of beta, error variance, and S/N ratio directly from the equations for the Continuous-continuous case as given on Phadke page 115.

$$\beta_{\text{OFF}_{\text{row}}} := \frac{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 \left(R_{\text{OFF}_{\text{row}}} \right)_{\text{signal, noise}} \cdot \text{signal_levels}_{\text{signal}}}{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 \left(\text{signal_levels}_{\text{signal}} \right)^2}$$

$$\sigma_{\text{eOFF}_{\text{row}}}^2 := \frac{1}{3 \cdot 3 - 1} \cdot \sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 \left[\left(R_{\text{OFF}_{\text{row}}} \right)_{\text{signal, noise}} - \beta_{\text{OFF}_{\text{row}}} \cdot \text{signal_levels}_{\text{signal}} \right]^2$$

$$\eta_{\text{OFF}_{\text{row}}} := 10 \cdot \log \left[\frac{\left(\beta_{\text{OFF}_{\text{row}}} \right)^2}{\sigma_{\text{eOFF}_{\text{row}}}^2} \right]$$

Define a function that will perform ANOVA for any response at any factor at all levels. The first argument is the name of the data set to be analyzed. The second argument is the number of the control factor. This will be helpful in creating the factor effect plots below.

```

m(response, factor_no) :=
  for level ∈ 1..3
  |
  | sum ← 0
  | for row ∈ 1..18
  |   sum ← sum + response_row if L18_row, factor_no = level
  | mean_level ←  $\frac{\text{sum}}{6}$ 
  |
  | mean

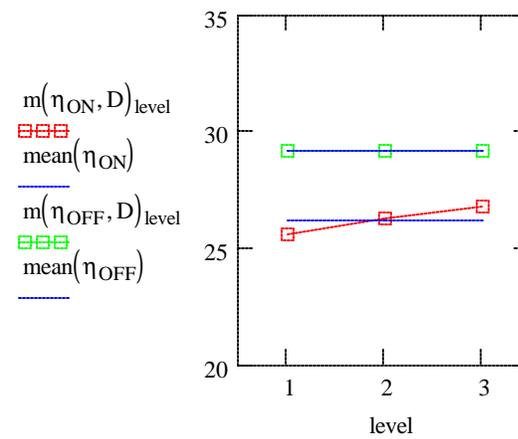
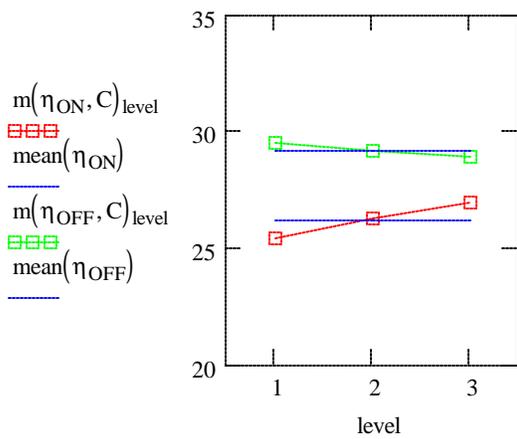
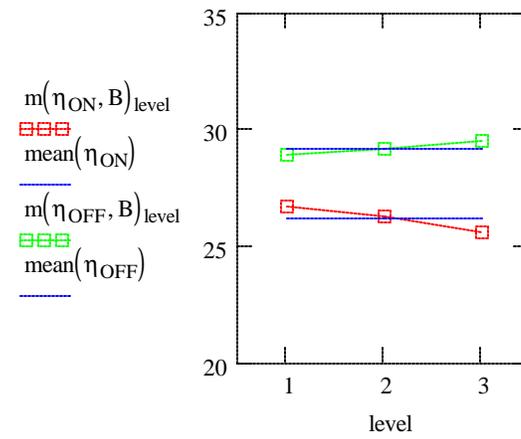
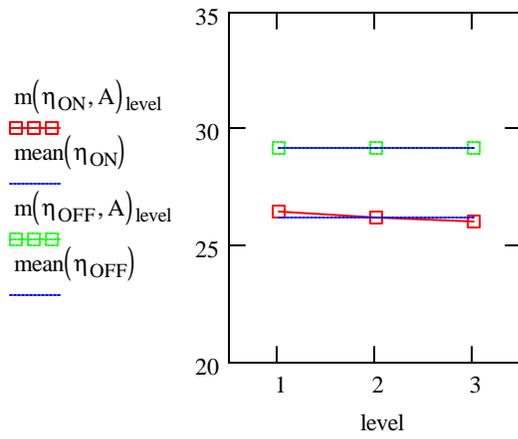
```

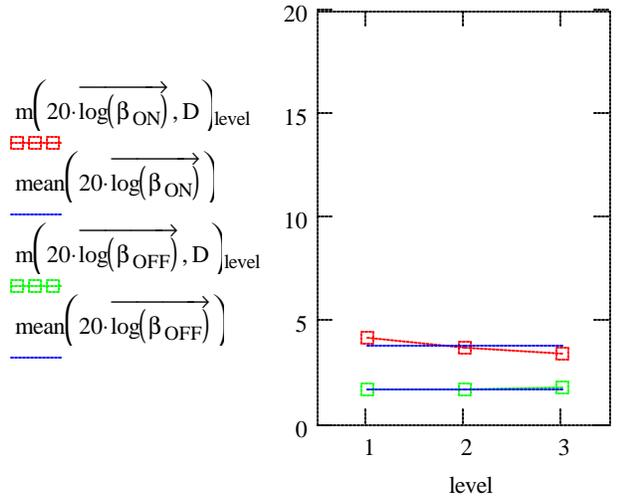
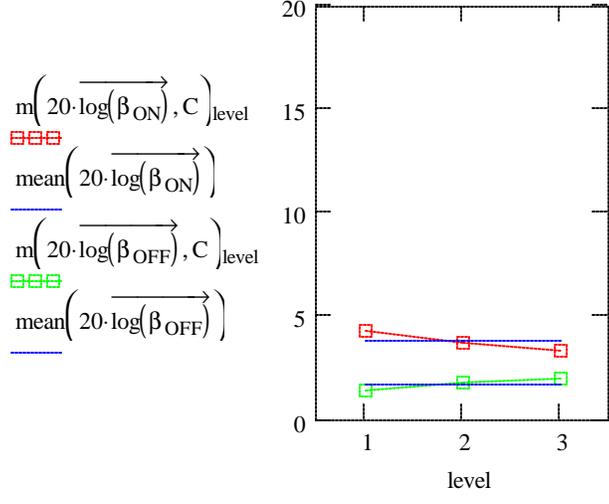
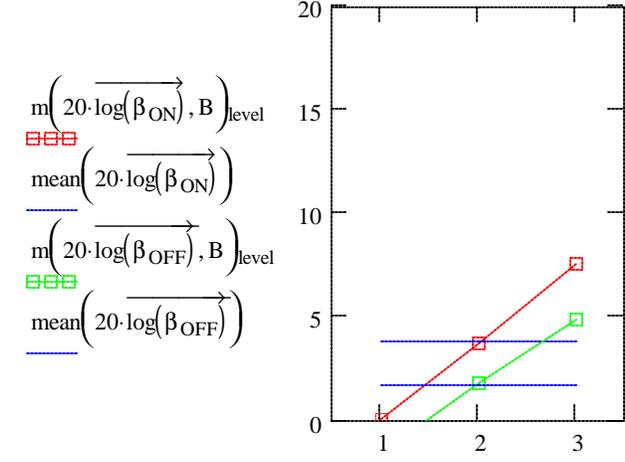
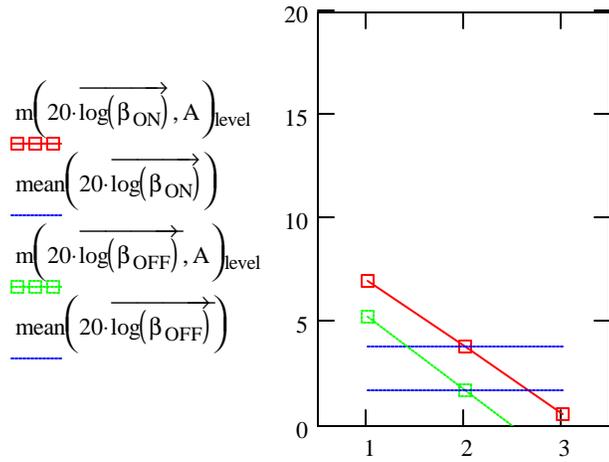
Define the range variable to step through the three level factors.

level := 1..3

Create a shorthand notation for the control factors consistent with Table 9.4.

A := 3 B := 4 C := 5 D := 7





Optimization

The optimum settings for this second iteration seem to be:

A=2 B=3 C=1 D=3

The optimum settings chosen by Padke on the FIRST optimization run were:

A=2 B=1 C=3 D=3

I'd say there's no point in reversing the values of B and C. Let's just leave them alone and change only D. So, my final decision on this round is to select

A=2 B=2 C=2 D=3

For RT-ON he prediction is:

$$\eta_{\text{pred2}} := -3 \cdot \text{mean}(\eta_{\text{ON}}) + m(\eta_{\text{ON},A})_2 + m(\eta_{\text{ON},B})_2 + m(\eta_{\text{ON},C})_2 + m(\eta_{\text{ON},D})_3$$

$$\eta_{\text{pred2}} = 26.885$$

Oddly, this is not as high as our last prediction, but is at least higher than our last realized value of S/N.

$$\eta_{\text{pred}} = 27.701$$

$$\eta_{\text{ON_OPT}} = 26.436$$

Confirmation

The true value is:

Run through all the signal and noise factors at the optimal control factor settings.

$$\text{RON_OPT} := \left[\begin{array}{l} \text{for signal} \in 1..3 \\ \text{for noise} \in 1..3 \\ y_{\text{signal, noise}} \leftarrow \text{RT_ON2} \left[\begin{array}{l} \text{signal_levels}_{\text{signal}}, \text{noise_levels}_{\text{noise}} \end{array} \right], \left(\begin{array}{l} 4.0 \\ 5.336 \\ 60.0 \\ 7.2 \cdot 1.2 \end{array} \right) \\ y \end{array} \right]$$

$$\text{RON_OPT} = \begin{pmatrix} 0.694 & 0.729 & 0.765 \\ 1.389 & 1.457 & 1.53 \\ 2.083 & 2.186 & 2.296 \end{pmatrix}$$

Here is the data you'd gather.

$$\beta_{\text{ON_OPT}} := \frac{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 (\text{RON_OPT})_{\text{signal, noise}} \cdot \text{signal_levels}_{\text{signal}}}{\sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 (\text{signal_levels}_{\text{signal}})^2}$$

$$\sigma_{eON_OPT}^2 := \frac{1}{3.3 - 1} \cdot \sum_{\text{signal} = 1}^3 \sum_{\text{noise} = 1}^3 \left[(R_{ON_OPT})_{\text{signal, noise}} - \beta_{ON_OPT} \cdot \text{signal_levels}_{\text{signal}} \right]^2$$

$$\eta_{ON_OPT} := 10 \cdot \log \left[\frac{(\beta_{ON_OPT})^2}{\sigma_{eON_OPT}^2} \right]$$

$$\eta_{ON_OPT} = 26.852$$

$$\eta_{\text{pred2}} = 26.885$$

Not a bad prediction. The forecast was for an increase of about 0.7 dB and we actually realized an increase of almost 0.7 dB.

This represents an improvement over our last realized value of S/N which was 26.43. However, the rate of increase in S/N is dropping quickly as we approach an optimal point.

D)₃