Lecture #9

Virtual Work

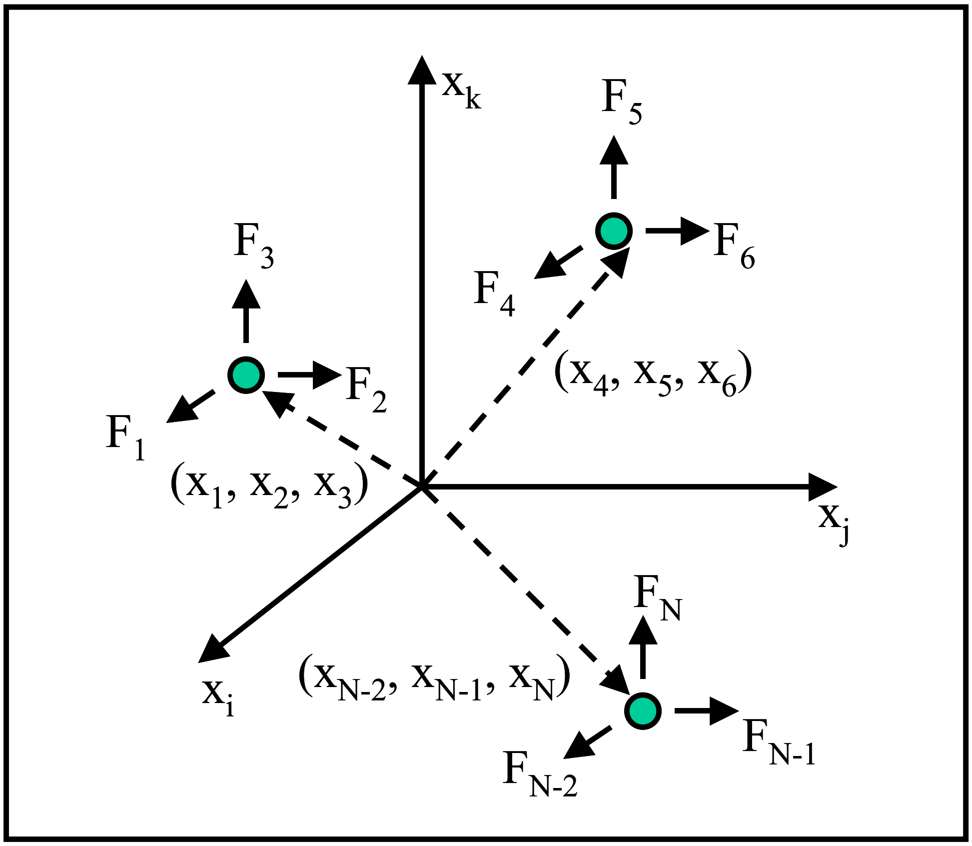
And the

Derivation of Lagrange's Equations

Derivation of Lagrangian Equations

Basic Concept: Virtual Work

Consider system of N particles located at $\left(x_1, x_2, x_3, \ldots x_{3N}\right)$ with 3 forces per particle $\left(F_1, F_2, F_3, \ldots F_{3N}\right)$, each in the positive direction.



Assume system given small, arbitrary displacements in all directions.

Called *virtual displacements*

- No passage of time
- Applied forces remain constant

The work done by the forces is termed **Virtual Work**.

$$\delta W = \sum_{j=1}^{3N} F_j \delta x_j$$

Note use of δx and not dx.

Note:

- There is no passage of time
- The forces remain constant.

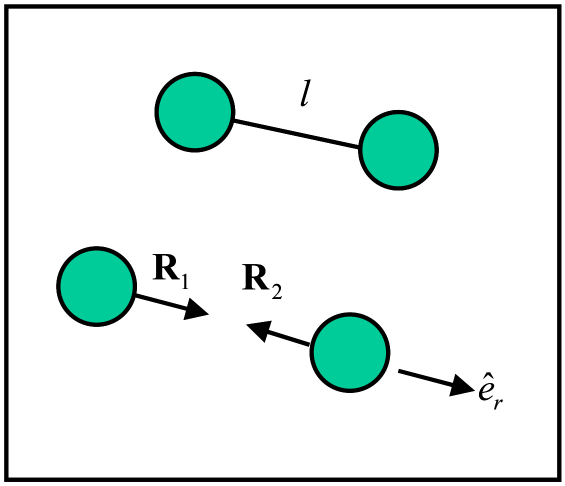
In vector form:

$$\delta W = \sum_{i=1}^{3} \mathbf{F}_i \bullet \delta \mathbf{r}_i$$

Virtual displacements MUST satisfy all constraint relationships,

➔ **Constraint forces do no work.**

Example: Two masses connected by a rod



Constraint forces:

$$\mathbf{R}_1 = -\mathbf{R}_2 = -R_2 \hat{e}_r$$

Now assume virtual displacements $\delta \mathbf{r}_1$, and $\delta \mathbf{r}_2$ - but the displacement components along the rigid rod must be equal, so there is a constraint equation of the form

$$e_r \bullet \delta r_1 = e_r \bullet \delta r_2$$

Virtual Work:

$$\begin{aligned} \delta W &= \mathbf{R}_1 \bullet \delta \mathbf{r}_1 + \mathbf{R}_2 \bullet \delta \mathbf{r}_2 \\ &= -R_2 \hat{e}_r \bullet \delta \mathbf{r}_1 + R_2 \hat{e}_r \bullet \delta \mathbf{r}_2 \\ &= \left(R_2 - R_2 \right) \hat{e}_r \bullet \delta \mathbf{r}_1 \\ &= 0 \end{aligned}$$

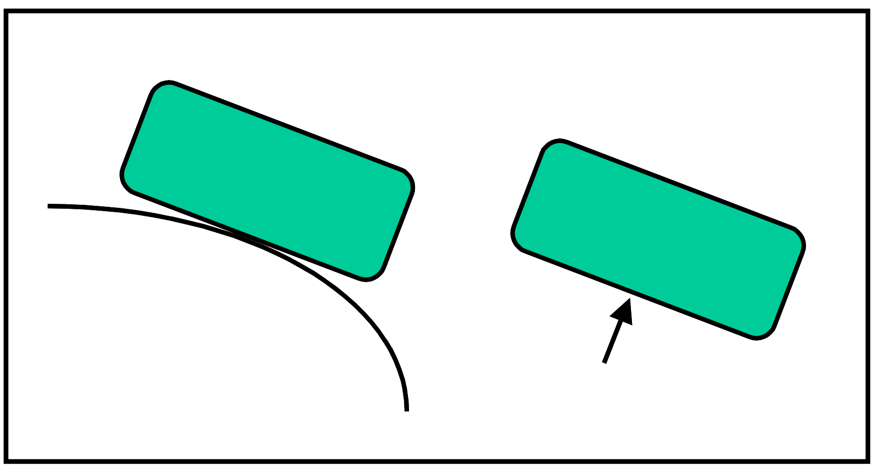
So the virtual work done of the constraint forces is zero

This analysis extends to rigid body case

- Rigid body is a collection of masses
- Masses held at a fixed distance.

➔ Virtual work for the internal constraints of a rigid body displacement is zero.

Example: Body sliding on rigid surface without friction



Since the surface is rigid and fixed, $\delta r_s = 0, \rightarrow \delta W = 0$

For the body, $\delta W = \mathbf{R}_1 \bullet \delta \mathbf{r}_1$, but the direction of the virtual displacement that *satisfies the constraints* is perpendicular to the constraint force. Thus $\delta W = 0$.

Principle of Virtual Work

m_i = mass of particle i

$\mathbf{R}_i$ = Constraint forces acting on the particle

$\mathbf{F}_i$ = External forces acting on the particle

➔ For static equilibrium (if all particles of the system are motionless in the inertial frame and if the vector sum of all forces acting on each particle is zero)

$$\mathbf{R}_i + \mathbf{F}_i = 0$$

The virtual work for a system in static equilibrium is

$$\delta W = \sum_{i=1}^{N} \left(\mathbf{R}_i + \mathbf{F}_i \right) \bullet \delta \mathbf{r}_i = 0$$

But virtual displacements must be perpendicular to constraint forces, so

$$\mathbf{R}_i \bullet \delta \mathbf{r}_i = 0 ,$$

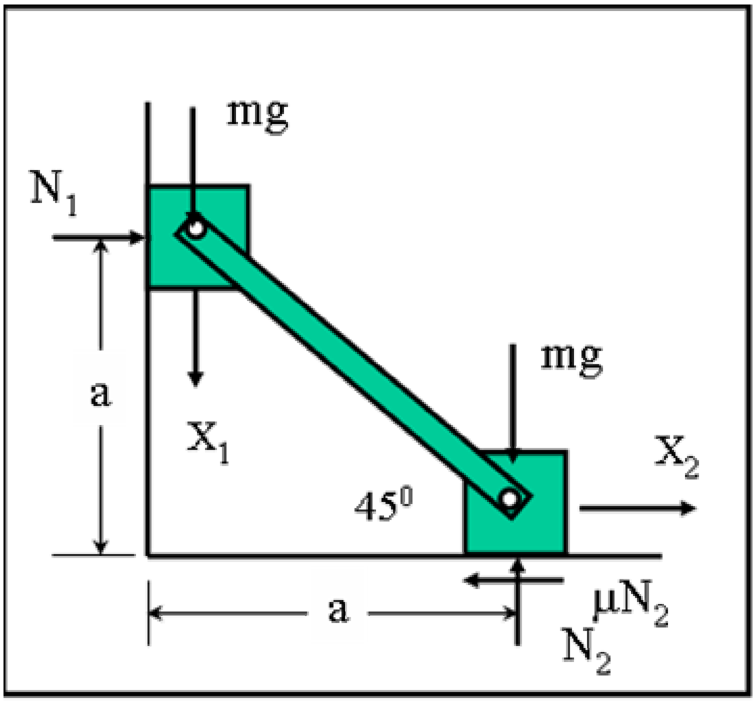
which implies that we have

$$\sum_{i=1}^{N} \mathbf{F}_i \bullet \delta \mathbf{r}_i = 0$$

Principle of virtual work:

The necessary and sufficient conditions for the static equilibrium of an initially motionless scleronomic system which is subject to workless bilateral constraints is that **zero virtual work** be done by the applied forces in moving through an arbitrary virtual displacement satisfying the constraints.

Example: System shown consists of 2 masses connected by a massless bar. Determine the coefficient of friction on the floor necessary for static equilibrium. (Wall is frictionless.)



Virtual Work: $\delta W = mg\delta x_1 - \mu N_2 \delta x_2$

Constraints and force balance: $\delta x_1 = \delta x_2, \quad N_2 = 2mg$

Substitution: $mg(1-2\mu)\delta x = 0$

Result: $\mu = \frac{1}{2}$

So far we have approached this as a statics problem, but this is a dynamics course!!

Recall d'Alembert who made dynamics a special case of statics:

$$\delta W = \sum_{i=1}^{N} \left(\mathbf{R}_i + \mathbf{F}_i - m_i \ddot{\mathbf{r}}_\mathrm{i} \right) \bullet \delta \mathbf{r}_i = 0$$

$$\Rightarrow \sum_{i=1}^{N} \left(\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i \right) \bullet \delta \mathbf{r}_i = 0$$

➔ So we can apply all of the previous results to the dynamics problem as well.

Comments:

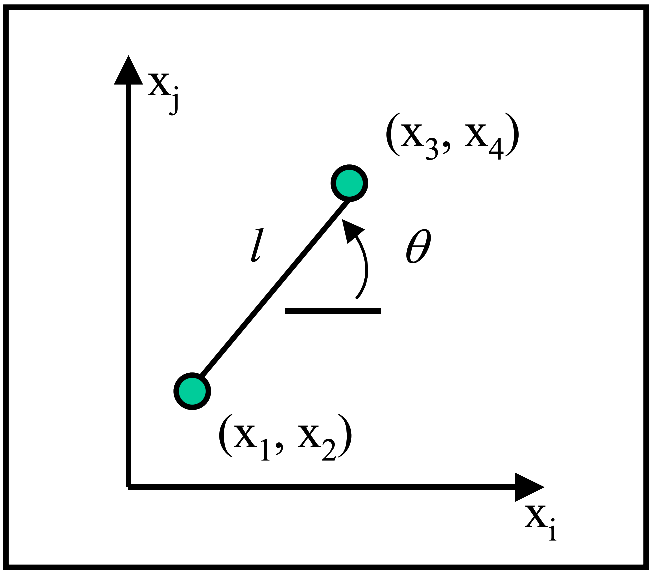
- Virtual work and virtual displacements play an important role in analytical dynamics, but fade from the picture in the application of the methods.

- However, this is why we can ignore the calculation of the constraint forces.

Generalized Forces

Since we have defined generalized coordinates, we need generalized forces to work in the same "space."

Consider the 2 particle problem:



Coordinates: x_1, x_2, x_3, x_4

Constraint: $(x_1 - x_3)^2 + (x_2 - x_4)^2 = l^2$

DOF: $4 - 1 = 3$

- Select $n=3$ generalized coordinates:

$$q_1 = \frac{(x_1 + x_3)}{2} \quad q_2 = \frac{(x_2 + x_4)}{2} \quad q_3 = \tan^{-1}\frac{(x_4 - x_2)}{(x_3 - x_1)}$$

- Can also write the inverse mapping:

$$x_i = f_i(q_1, q_2, q_3, \ldots q_n, t)$$

Virtual Work: $$\delta W = \sum_{j=1}^{3N} F_j \delta x_j$$

Constraint relations: $$\delta x_j = \sum_{i=1}^{n} \left(\frac{\partial x_j}{\partial q_i} \right) \delta q_i$$

Substitution: $$\delta W = \sum_{j=1}^{3N} \sum_{i=1}^{n} F_j \left(\frac{\partial x_j}{\partial q_i} \right) \delta q_i$$

Define Generalized Force: $$Q_i = \sum_{j=1}^{3N} F_j \left(\frac{\partial x_j}{\partial q_i} \right)$$

➔ Work done for unit displacement of q_i by forces acting on the system when all other generalized coordinates remain constant.

$$\Rightarrow \delta W = \sum_{i=1}^{n} Q_i \delta q_i$$

- If q_i is an angle, Q_i is a torque
- If q_i is a length, Q_i is a force
- If the q_i's are independent, then for static equilibrium must have:

$$Q_i = 0, \quad i = 1, 2, \ldots n$$

Derivation of Lagrange's Equation

- Two approaches

(A) Start with energy expressions

Formulation

Lagrange's Equations (Greenwood, 6-6)

Interpretation

Newton's Laws

(B) Start with Newton's Laws

Formulation

Lagrange's Equations (Wells, Chapters 3&4)

Interpretation

Energy Expressions

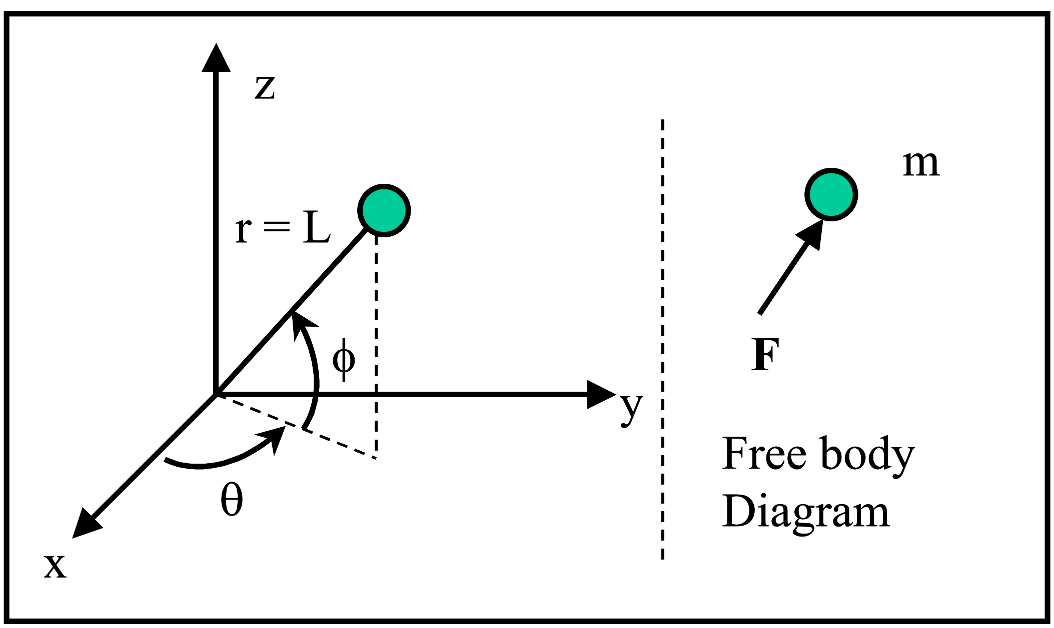
(A) Replicated the application of Lagrange's equations in solving problems

(B) Provides more insight and feel for the physics

Our process

1. Start with Newton
2. Apply virtual work
3. Introduce generalized coordinates
4. Eliminate constraints
5. Using definition of derivatives, eliminate explicit use of acceleration

- Start with a single particle with a single constraint, *e.g.*
    - Marble rolling on a frictionless sphere,
    - Conical pendulum



1. **Newton**: $\mathbf{F} = m\mathbf{a}$

- For the particle: $F_x = m\ddot{x}, F_y = m\ddot{y}, F_z = m\ddot{z}$ where axes *x, y, z* describe an inertial frame

- Note that F_x, F_y, F_z are the vector sum of all forces acting on the particle (applied and constraint forces)

2. **Apply Virtual Work**:

- Consider δ**s,** which is an arbitrary displacement for the system, then the virtual work associated with this displacement is:

$$\delta W = F_x \delta x + F_y \delta y + F_z \delta z$$

- Note that δ**s** may violate the applied constraints, because **F** contains constraint forces

- Combine Newton and Virtual Work

$$F_x \delta x = m\ddot{x}\delta x$$
$$F_y \delta y = m\ddot{y}\delta y$$
$$F_z \delta z = m\ddot{z}\delta z$$

- Add the equations

$$m\left(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z\right) = F_x \delta x + F_y \delta y + F_z \delta z$$

- Called D'Alembert's equation:

$$m\left(\ddot{x}\delta x+\ddot{y}\delta y+\ddot{z}\delta z\right)=F_x\delta x+F_y\delta y+F_z\delta z$$

- Observations:
  - Scalar relationship
  - LHS ≈ kinetic energy
  - RHS ≈ virtual work term

3. **Introduce generalized coordinates**
   - Assumed motion on a sphere ➔ 1 stationary constraint
   - DOF = 3 - 1 = 2 generalized coordinates

$$x=f_1\left(q_1,q_2\right),\quad y=f_2\left(q_1,q_2\right),\quad z=f_3\left(q_1,q_2\right)$$

- Define virtual displacements in terms of generalized coordinates:

$$\delta x_j=\sum_{i=1}^{n}\left(\frac{\partial x_j}{\partial q_i}\right)\delta q_i \quad ➔ \quad \begin{aligned}\delta x&=\frac{\partial x}{\partial q_1}\delta q_1+\frac{\partial x}{\partial q_2}\delta q_2\\ \delta y&=\frac{\partial y}{\partial q_1}\delta q_1+\frac{\partial y}{\partial q_2}\delta q_2\\ \delta z&=\frac{\partial z}{\partial q_1}\delta q_1+\frac{\partial z}{\partial q_1}\delta q_2\end{aligned}$$

- **Note:** these virtual displacements conform to the constraints, because the mapping of the generalized coordinates conforms to the surface of the sphere.

- Substitute virtual displacements into D'Alembert's equation

$$\delta x = \frac{\partial x}{\partial q_1}\delta q_1 + \frac{\partial x}{\partial q_2}\delta q_2$$
$$\delta y = \frac{\partial y}{\partial q_1}\delta q_1 + \frac{\partial y}{\partial q_2}\delta q_2$$
$$\delta z = \frac{\partial z}{\partial q_1}\delta q_1 + \frac{\partial z}{\partial q_1}\delta q_2$$

$$\Downarrow$$

$$m\left(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z\right) = F_x\delta x + F_y\delta y + F_z\delta z$$

$$\Downarrow$$

$$m\left(\ddot{x}\frac{\partial x}{\partial q_1} + \ddot{y}\frac{\partial y}{\partial q_1} + \ddot{z}\frac{\partial z}{\partial q_1}\right)\delta q_1 + m\left(\ddot{x}\frac{\partial x}{\partial q_2} + \ddot{y}\frac{\partial y}{\partial q_2} + \ddot{z}\frac{\partial z}{\partial q_2}\right)\delta q_2$$
$$= \left(F_x\frac{\partial x}{\partial q_1} + F_y\frac{\partial y}{\partial q_1} + F_z\frac{\partial z}{\partial q_1}\right)\delta q_1 + \left(F_x\frac{\partial x}{\partial q_2} + F_y\frac{\partial y}{\partial q_2} + F_z\frac{\partial z}{\partial q_2}\right)\delta q_2$$

- Facts:
  - Virtual displacements δq_1 and δq_2 conform to constraints
  - Virtual work δW is work that conforms to constraints
  - δq_1 and δq_2 are independent and can be independently moved without violating constraints

- **Conclusion:**
  - Force of the constraint has been eliminated by selecting generalized coordinates that enforce the constraint (Reason 2 for Lagrange, pg 24)
  - Further, we can split the equation into two equations in two unknowns due to independence of δq_1 and δq_2.

$$m\left(\ddot{x}\frac{\partial x}{\partial q_1}+\ddot{y}\frac{\partial y}{\partial q_1}+\ddot{z}\frac{\partial z}{\partial q_1}\right)=\left(F_x\frac{\partial x}{\partial q_1}+F_y\frac{\partial y}{\partial q_1}+F_z\frac{\partial z}{\partial q_1}\right)$$

$$m\left(\ddot{x}\frac{\partial x}{\partial q_2}+\ddot{y}\frac{\partial y}{\partial q_2}+\ddot{z}\frac{\partial z}{\partial q_2}\right)=\left(F_x\frac{\partial x}{\partial q_2}+F_y\frac{\partial y}{\partial q_2}+F_z\frac{\partial z}{\partial q_2}\right)$$

5. **Finally, eliminate acceleration terms**

- Consider the total derivative

$$\frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1}\right)=\ddot{x}\frac{\partial x}{\partial q_1}+\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right)$$

- Rearrange

$$\ddot{x}\frac{\partial x}{\partial q_1}=\frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1}\right)-\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) \qquad (1)$$

- Recall

$$x = f_1(q_1, q_2) \quad \therefore \quad \dot{x} = \frac{d}{dt}\left[f_1(q_1, q_2)\right]$$

- Perform the derivative (chain rule):

$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 \tag{2}$$

- Partial derivative of (2) with respect to $\dot{q}_1$ gives

$$\frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial x}{\partial q_1} \tag{3}$$

- Since $x = f_1(q_1, q_2)$, $\frac{\partial x}{\partial q_1} = g_1(q_1, q_2)$ is a ftn of both q_1 and q_2 the time derivative of $\frac{\partial x}{\partial q_1}$ gives (chain rule again)

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{\partial}{\partial q_1}\left(\frac{\partial x}{\partial q_1}\right)\dot{q}_1 + \frac{\partial}{\partial q_2}\left(\frac{\partial x}{\partial q_1}\right)\dot{q}_2 \tag{4}$$

- Partial derivative of $\dot{x}$ (2) with respect to q_1 gives

$$\frac{\partial \dot{x}}{\partial q_1} = \frac{\partial}{\partial q_1}\left(\frac{\partial x}{\partial q_1}\right)\dot{q}_1 + \frac{\partial}{\partial q_1}\left(\frac{\partial x}{\partial q_2}\right)\dot{q}_2 \tag{5}$$

- Note RHS of 4 and 5 are the same, thus

$$\Rightarrow \quad \frac{\partial \dot{x}}{\partial q_1} = \frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) \tag{6}$$

- Now, insert (3) and (6) into (1):

$$\ddot{x}\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1}\right) - \dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) \tag{1}$$

$$\frac{\partial x}{\partial q_1} = \frac{\partial \dot{x}}{\partial \dot{q}_1} \qquad \frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{\partial \dot{x}}{\partial q_1} \tag{3 and 6}$$

- Results in:

$$\ddot{x}\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_1}\right) - \dot{x}\frac{\partial \dot{x}}{\partial q_1} \tag{7}$$

- Note that

$$\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial\left(\frac{\dot{x}^2}{2}\right)}{\partial \dot{q}_1} \quad \text{and} \quad \dot{x}\frac{\partial \dot{x}}{\partial q_1} = \frac{\partial\left(\frac{\dot{x}^2}{2}\right)}{\partial q_1}$$

- Finally

$$\ddot{x}\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\frac{\partial\left(\frac{\dot{x}^2}{2}\right)}{\partial \dot{q}_1}\right) - \frac{\partial\left(\frac{\dot{x}^2}{2}\right)}{\partial q_1} \tag{8}$$

- The above process is identical for *y* and *z*.

- Recall our virtual work equation for q_1:

$$m\left(\ddot{x}\frac{\partial x}{\partial q_1} + \ddot{y}\frac{\partial y}{\partial q_1} + \ddot{z}\frac{\partial z}{\partial q_1}\right) = \left(F_x\frac{\partial x}{\partial q_1} + F_y\frac{\partial y}{\partial q_1} + F_z\frac{\partial z}{\partial q_1}\right)$$

- Insert equations (8) for *x, y* and *z* and collect terms to eliminate acceleration terms. (Reason 3 for Lagrange, pg 24)

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_1}\left(m\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2}\right)\right) - \frac{\partial}{\partial q_1}\left(m\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2}\right)$$
$$= \left(F_x\frac{\partial x}{\partial q_1} + F_y\frac{\partial y}{\partial q_1} + F_z\frac{\partial z}{\partial q_1}\right)$$

- Observe that:

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

which is the **kinetic energy of the particle**

- Finally:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_1} T \right) - \frac{\partial}{\partial q_1} T = \left(F_x \frac{\partial x}{\partial q_1} + F_y \frac{\partial y}{\partial q_1} + F_z \frac{\partial z}{\partial q_1} \right)$$

- Similarly:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_2} T \right) - \frac{\partial}{\partial q_2} T = \left(F_x \frac{\partial x}{\partial q_2} + F_y \frac{\partial y}{\partial q_2} + F_z \frac{\partial z}{\partial q_2} \right)$$

- The general form of Lagrange's equation is thus:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_r} T \right) - \frac{\partial}{\partial q_r} T = Q_{q_r}$$

$$Q_{q_r} = \left(F_x \frac{\partial x}{\partial q_r} + F_y \frac{\partial y}{\partial q_r} + F_z \frac{\partial z}{\partial q_r} \right)$$

- Some observations:
  - One Lagrange equation needed for each DOF
  - Easily extendable for a system of particles
  - T – Expression of system kinetic energy
  - All inertial forces contained in the LHS
  - Q_{q_r} only contains external forces
- **How to use this ….**

1. Determine number of DOF and constraints

2. Identify generalized coordinates and equations of constraint

   a. Iterate on 1 and 2 if needed

3. Write expression for T

   a. *v* inertial velocity that can be written in terms of the coordinates of any frame

   b. Find required derivatives of T

4. Find generalized forces Q_{q_r}

   a. If forces are known in inertial coordinates, transform them to generalized coordinates

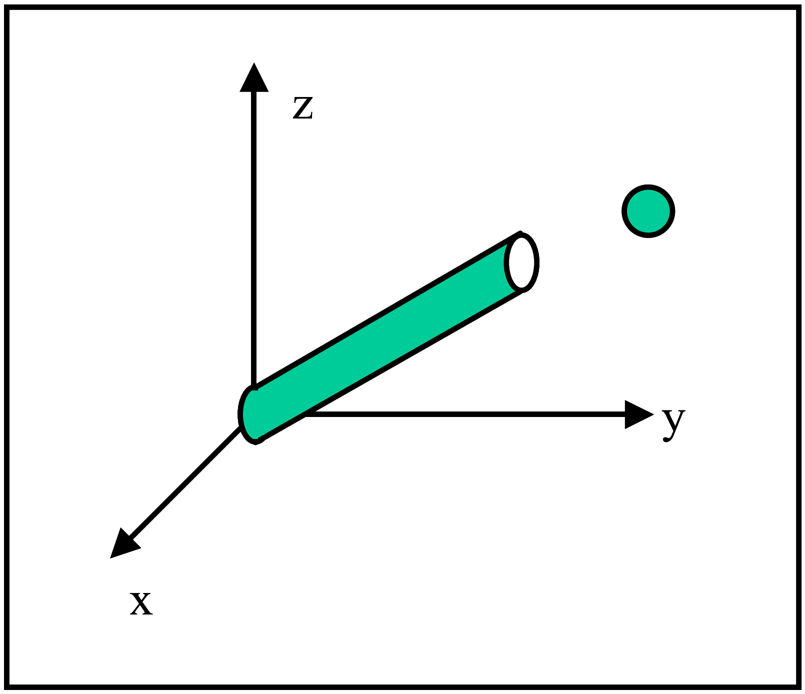
   b. Apply generalized force equation for each force

$$Q_{q_r} = \left(F_x \frac{\partial x}{\partial q_r} + F_y \frac{\partial y}{\partial q_r} + F_z \frac{\partial z}{\partial q_r} \right)$$

5. Substitute into Lagrange's equation

6. Solve analytically or numerically

Example: Projectile Problem:



1, 2, 3. DOF = 3, no constraints

4. $$T = \frac{1}{2} m v^2 = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

5. $$\frac{\partial T}{\partial \dot{x}} = m\dot{x},\ \frac{\partial T}{\partial \dot{y}} = m\dot{y},\ \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x},\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = m\ddot{y},\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = m\ddot{z}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0$$

6. Generalized forces:

$$Q_{q_r} = \left(F_x \frac{\partial x}{\partial q_r} + F_y \frac{\partial y}{\partial q_r} + F_z \frac{\partial z}{\partial q_r} \right)$$

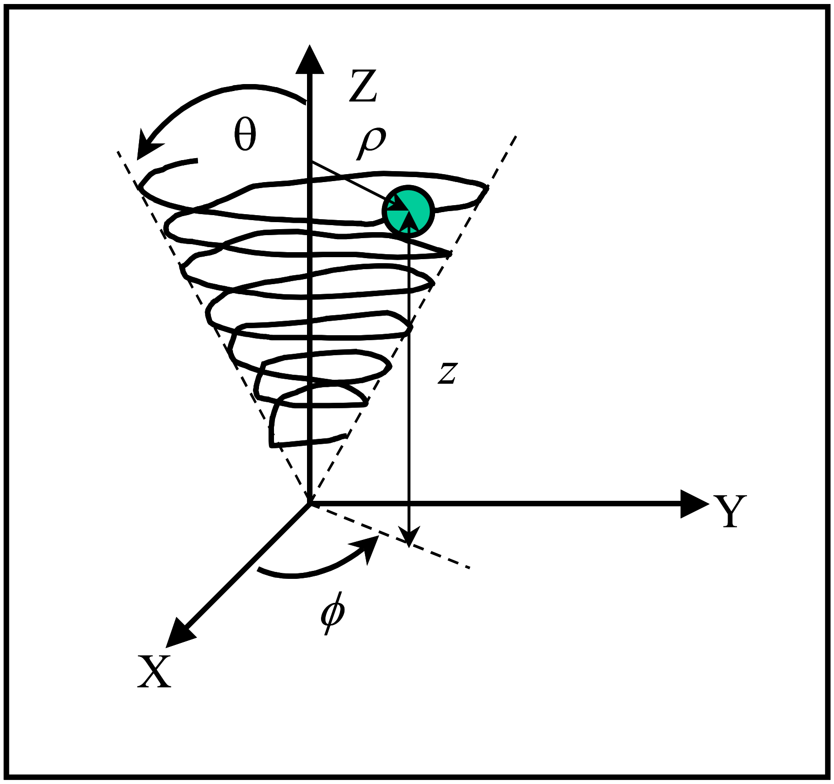
$$\mathbf{F} = -mg\,\hat{z}$$

$$Q_{q_x} = Q_{q_y} = 0, \quad Q_{q_z} = F_z \frac{\partial z}{\partial z} = -mg$$

7. EOMs: $m\ddot{x} = 0, = m\ddot{y} = 0, \; m\ddot{z} = -mg$
8. Solve differential equations.

- Comments:
  - Method is overkill for this problem
  - Inspection shows agreement with Newton

Example: Mass moving along a frictionless track.



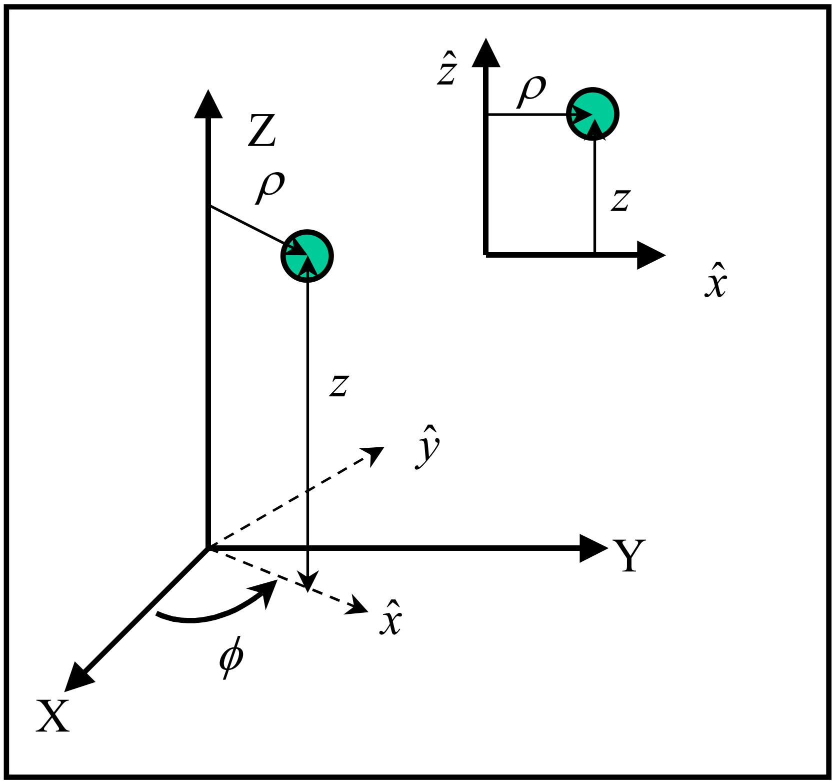
- Track geometry defined such that:

  $\rho = az$, and $\phi = -bz$

  DOF = 3 – 2 =1
- Constraint equations: $\rho = az$, and $\phi = -bz$
- Generalized coordinate: z

- Find $T = \frac{1}{2}mv^2$, what is v?

- Define rotating coordinate frame such that mass remains in $\hat{x}-\hat{z}$ plane.



$$r = \rho\hat{x} + z\hat{z} = az\hat{x} + z\hat{z}$$ and $$\omega = \dot{\phi}\hat{z} = -bz\hat{z}$$

$$\begin{aligned}\dot{r} &= a\dot{z}\hat{x} + \dot{z}\hat{z} + (-bz\hat{z}) \times az\hat{x} + z\hat{z} \\ &= a\dot{z}\hat{x} - ab\dot{z}z\hat{y} + \dot{z}\hat{z}\end{aligned}$$

➔

$$\begin{aligned}v^2 &= \dot{r} \bullet \dot{r} \\ &= (a\dot{z})^2 + (ab\dot{z}z)^2 + \dot{z}^2 \\ &= (1 + a^2 + a^2b^2z^2)\dot{z}^2\end{aligned}$$

$$T = \frac{m}{2}(1 + a^2 + a^2b^2z^2)\dot{z}^2$$ and $$\frac{\partial T}{\partial \dot{z}} = m(1 + a^2 + a^2b^2z^2)\dot{z}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = m(1 + a^2 + a^2b^2z^2)\ddot{z} + 2m\left(a^2b^2z\right)\dot{z}^2$$

$$\frac{\partial T}{\partial z} = m(a^2b^2z)\dot{z}^2$$

- External force is gravity

$$Q_{q_r} = \left(F_x\frac{\partial x}{\partial q_r} + F_y\frac{\partial y}{\partial q_r} + F_z\frac{\partial z}{\partial q_r}\right)$$

$$\mathbf{F} = -mg\,\hat{z}$$

$$Q_{q_x} = Q_{q_y} = 0, \quad Q_{q_z} = F_z\frac{\partial z}{\partial z} = -mg$$

- Equation of Motion:

$$\left(a^2 + a^2b^2z^2 + 1\right)\ddot{z} + a^2b^2z\,\dot{z}^2 = -g$$

- Comments:
  - o Solution highly nonlinear
  - o "Trick" was finding inertial velocity
  - o Still need to use FARM approach

Extending Lagrange's Equation to Systems with Multiple Particles

- Assume a system of particles and apply Newton's laws:

$$\begin{gathered} F_{x_1} = m\ddot{x}_1,\ F_{y_1} = m\ddot{y}_1,\ F_{z_1} = m\ddot{z}_1 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ F_{x_p} = m\ddot{x}_p,\ F_{y_p} = m\ddot{y}_p,\ F_{z_p} = m\ddot{z}_p \end{gathered}$$

- As before, the *F*'s contain both external and constraint forces.
- Multiply both sides of each equation by the appropriate virtual displacement and add all the equations together.

$$\sum_{i=1}^{p} m\left(\ddot{x}_i\delta x_i + \ddot{y}_i\delta y_i + \ddot{z}_i\delta z_i\right) = \sum_{i=1}^{p}\left(F_{x_i}\delta x_i + F_{y_i}\delta y_i + F_{z_i}\delta z_i\right)$$

- Recall that this is D'Alembert's equation
- Assume the system has *n* DOF, $n \le 3p$
- Select generalized coordinates, q_i that enforce the constraints:

$$\begin{aligned} x_i &= f_i\left(q_1, q_2, \ldots, q_n, t\right) \\ y_i &= g_i\left(q_1, q_2, \ldots, q_n, t\right) \\ z_i &= h_i\left(q_1, q_2, \ldots, q_n, t\right) \end{aligned}$$

- Express virtual displacements in terms of generalized coordinates:

$$\delta x_i = \frac{\partial x_i}{\partial q_1}\delta q_1 + \frac{\partial x_i}{\partial q_2}\delta q_2 + \ldots + \frac{\partial x_i}{\partial q_n}\delta q_n$$

$$\delta y_i = \frac{\partial y_i}{\partial q_1}\delta q_1 + \frac{\partial y_i}{\partial q_2}\delta q_2 + \ldots + \frac{\partial y_i}{\partial q_n}\delta q_n$$

$$\delta z_i = \frac{\partial z_i}{\partial q_1}\delta q_1 + \frac{\partial z_i}{\partial q_2}\delta q_2 + \ldots + \frac{\partial z_i}{\partial q_n}\delta q_n$$

- Substitute the relations into D'Alembert's equation

$$\sum_{i=1}^{p} m\left(\ddot{x}_i \frac{\partial x_i}{\partial q_r} + \ddot{y}_i \frac{\partial y_i}{\partial q_r} + \ddot{z}_i \frac{\partial z_i}{\partial q_r}\right)\delta q_r$$

$$= \sum_{i=1}^{p}\left(F_{x_i}\frac{\partial x_i}{\partial q_r} + F_{y_i}\frac{\partial y_i}{\partial q_r} + F_{z_i}\frac{\partial z_i}{\partial q_r}\right)\delta q_r$$

- As before, have used fact that the generalized coordinates **automatically enforce the constraints**.
  - Sum over the entire system of particles decouples for each of the generalized coordinates.
  - This leaves us *n* such equations.

- Using the calculus relations (chain rule), one can show that

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_r}T\right) - \frac{\partial}{\partial q_r}T = Q_{q_r}$$

- Once again, one Lagrange equation for each DOF.

Lagrange's Equation for Conservative Systems

- Conservative forces and conservative systems
  - Forces are such that the work done by the forces in moving the system from one state to another depends only on the initial and final coordinates of the particles (path independence).

- **Potential Energy**, V
  - Work done by a conservative force in a transfer from a general configuration A to a reference configuration B is the potential energy of the system at A with respect to B.
  - **Note:** V is defined as work from the general state to the reference state.

- Examples of **conservative forces**:
  - Springs (linear elastic)
  - Elastic bodies
  - Gravity force

- **Non-conservative forces**
  - Friction
  - Drag of a fluid
  - Any force with time or velocity dependence

- General Expression for V, the potential energy

$$V = -\int_B^A \sum_{i=1}^{P} \left(F_{x_i} dx_i + F_{y_i} dy_i + F_{z_i} dz_i \right)$$

- Note the "–" sign since the path is from B to A. The sum is over the P particles in the system.

- For path independence, integrand must be an *exact differential*. Thus:

$$F_{x_i} = -\frac{\partial V}{\partial x_i} \quad F_{y_i} = -\frac{\partial V}{\partial y_i} \quad F_{z_i} = -\frac{\partial V}{\partial z_i} \qquad \text{(C1)}$$

- Observe that:

$$\frac{\partial F_{x_3}}{\partial y_4} = \frac{\partial}{\partial y_4}\left(-\frac{\partial V}{\partial x_3} \right) = -\frac{\partial^2 V}{\partial x_3 \partial y_4}$$

$$\frac{\partial F_{y_4}}{\partial x_3} = \frac{\partial}{\partial x_3}\left(-\frac{\partial V}{\partial y_4} \right) = -\frac{\partial^2 V}{\partial x_3 \partial y_4}$$

- Thus, in general

$$\frac{\partial F_{x_i}}{\partial y_r} = \frac{\partial F_{y_r}}{\partial x_i} \qquad \text{(C2)}$$

- Equation (C1) represents a necessary condition for a force to be conservative, Equation (C2) is a sufficient condition.

- Recall expression for generalized forces:

$$Q_{qr} = \sum_{i=1}^{p} \left(F_{x_i} \frac{\partial x_i}{\partial q_r} + F_{y_i} \frac{\partial y_i}{\partial q_r} + F_{z_i} \frac{\partial z_i}{\partial q_r} \right)$$

  - Separate forces into conservative and non-conservative

$$Q_{qr} = -\sum_{i=1}^{p} \left(\frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial q_r} + \frac{\partial V}{\partial y_i} \frac{\partial y_i}{\partial q_r} + \frac{\partial V}{\partial y_i} \frac{\partial z_i}{\partial q_r} \right) + Q^N{}_{qr}$$

$$= -\frac{\partial V}{\partial q_r} + Q^N{}_{qr}$$

- Lagrange's Equation:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_r} T \right) - \frac{\partial}{\partial q_r} T = Q_{q_r}$$

- Substitute in generalized force:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_r} T \right) - \frac{\partial}{\partial q_r} T = -\frac{\partial V}{\partial q_r} + Q^N{}_{qr}$$

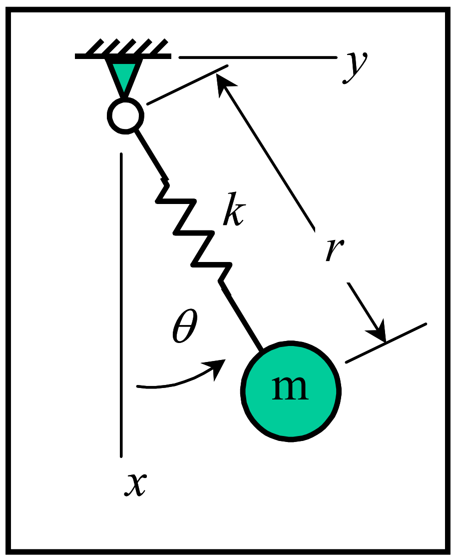
$$\Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_r} T \right) - \frac{\partial}{\partial q_r} (T - V) = Q^N{}_{qr}$$

- Since conservative forces are not functions of velocities: $\frac{\partial}{\partial \dot{q}_r} V = 0$

- Thus, can **define the Lagrangian** $L = T - V$ to obtain the final form of Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = F_{qr}$$

Example: Planar pendulum with an inline spring.



- DOF = 3 – 1 = 2
- Constraint equation: $z = 0$
- Generalized coordinates: $r,\ \theta$
- Coordinate mapping: $x = r\cos\theta, \quad y = r\sin\theta$

- Kinetic energy

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right)$$

- Derivatives of coordinates:

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

- Substitute into kinetic energy equation

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

- Potential energy

$$V = \frac{1}{2} k\left(r - r_o\right)^2 - mgr\cos\theta$$

- Lagrangian

$$L = T - V = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2} k\left(r - r_o\right)^2 + mgr\cos\theta$$

- Derivatives of L (note need to do this for each GC)

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}, \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - k\left(r - r_o\right) + mg\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = -mgr\sin\theta$$

- Substitute into Lagrange's Equation:

$$m\ddot{r} - mr\dot{\theta}^2 + k\left(r - r_o\right) = mg\cos\theta$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgr\sin\theta = 0$$