Lecture #7

Lagrange's Equations

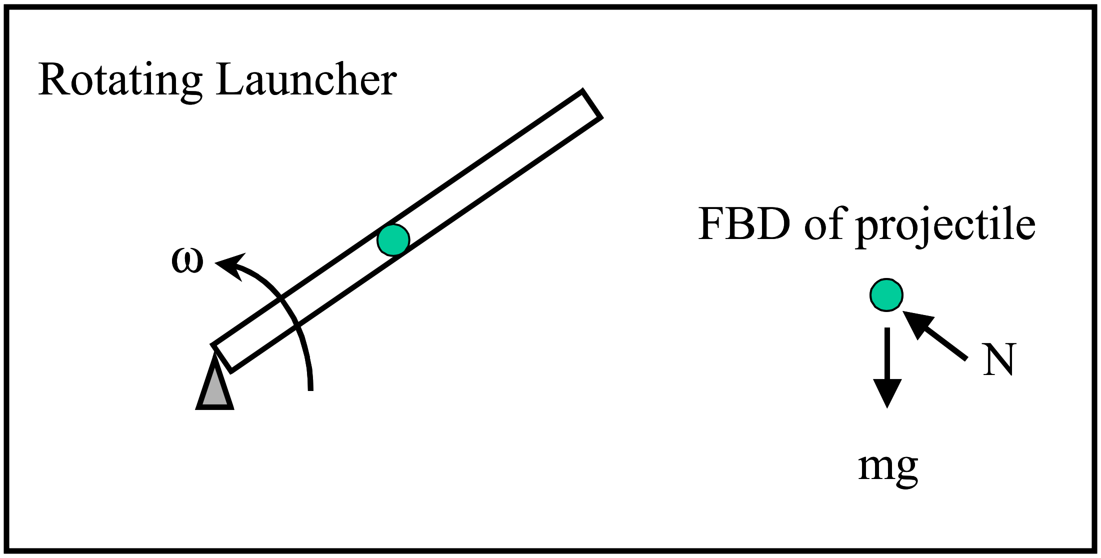
Lagrange's Equations

Joseph-Louis Lagrange 1736-1813

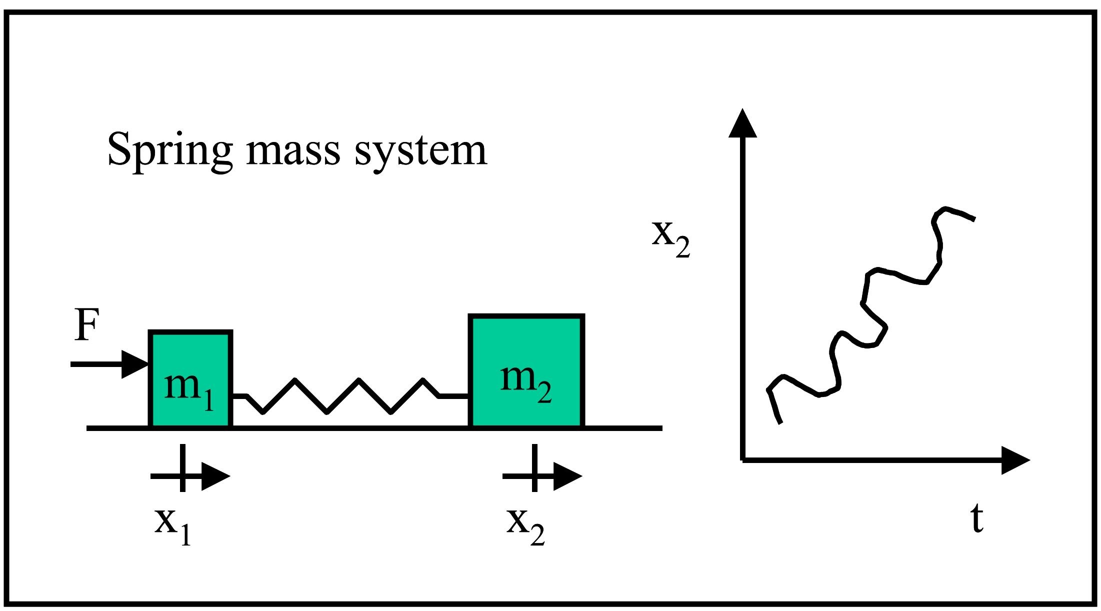
- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Lagrange.html
- Born in Italy, later lived in Berlin and Paris.
- Originally studied to be a lawyer
- Interest in math from reading Halley's 1693 work on algebra in optics
- "If I had been rich, I probably would not have devoted myself to mathematics."
- Contemporary of Euler, Bernoulli, Leibniz, D'Alembert, Laplace, Legendre (Newton 1643-1727)
- Contributions
  - Calculus of variations
  - Calculus of probabilities
  - Propagation of sound
  - Vibrating strings
  - Integration of differential equations
  - Orbits
  - Number theory
  - …
- "… whatever this great man says, deserves the highest degree of consideration, but he is too abstract for youth" -- student at *Ecole Polytechnique*.

Why Lagrange (or why NOT Newton)

- Newton – Given motion, deduce forces



- Or given forces – solve for motion



Great for "simple systems"

What about “real” systems? Complexity increased by:

- Vectoral equations – difficult to manage
- Constraints – what holds the system together?
- No general procedures

Lagrange provides:

- Avoiding <u>some</u> constraints
- Equations presented in a standard form

➔ Termed Analytic Mechanics

- Originated by Leibnitz (1646-1716)
- Motion (or equilibrium) is determined by <u>scalar</u> equations

<u>Big Picture</u>

- Use kinetic and potential energy to solve for the motion
- No need to solve for accelerations (KE is a velocity term)
- Do need to solve for **inertial** velocities

Let’s start with the answer, and then explain how we get there.

Define: **Lagrangian Function**

- L = T – V (Kinetic – Potential energies)

Lagrange's Equation

- For conservative systems

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0$$

- Results in the differential equations that describe the equations of motion of the system

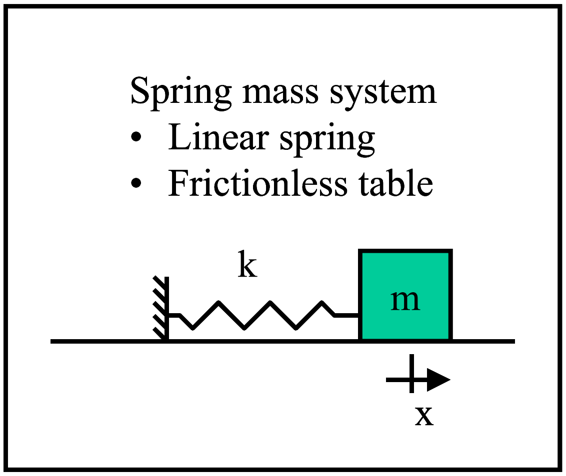
Key point:

- Newton approach requires that you find accelerations in all 3 directions, equate F=ma, solve for the constraint forces, and then eliminate these to reduce the problem to "characteristic size"
- Lagrangian approach enables us to immediately reduce the problem to this "characteristic size" ➔ we only have to solve for that many equations in the first place.

The ease of handling external constraints really differentiates the two approaches

Simple Example

- Spring – mass system



- Lagrangian L = T – V

$$L = T - V = \tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} kx^2$$

- Lagrange's Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

- Do the derivatives

$$\frac{\partial L}{\partial \dot{q}_i} = m\dot{x} \quad , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = m\ddot{x} \quad , \quad \frac{\partial L}{\partial q_i} = -kx$$

- Put it all together

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = m\ddot{x} + kx = 0$$

Consider the MGR problem with the mass oscillating between the two springs. Only 1 degree of freedom of interest here so, take $q_i = R$

$$\dot{r}_M^I = \begin{bmatrix} \dot{R} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}^{\times} \begin{bmatrix} R_o + R \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{R} \\ \omega(R_o + R) \\ 0 \end{bmatrix}$$

$$T = \frac{m}{2}(\dot{r}_M^I)^T(\dot{r}_M^I) = \frac{m}{2}\left(\dot{R}^2 + \omega^2(R_o + R)^2\right)$$

$$V = 2\frac{k}{2}R^2$$

$$L = T - V = \frac{m}{2}\left(\dot{R}^2 + \omega^2(R_o + R)^2\right) - kR^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) = m\ddot{R}$$

$$\frac{\partial L}{\partial R} = m\omega^2(R_o + R) - 2kR$$

So the equations of motion are: $m\ddot{R} - m\omega^2(R_o + R) + 2kR = 0$

or $\ddot{R} + \left(\frac{2k}{m} - \omega^2\right)R = R_o\omega^2$ which is the same as on (3-4).

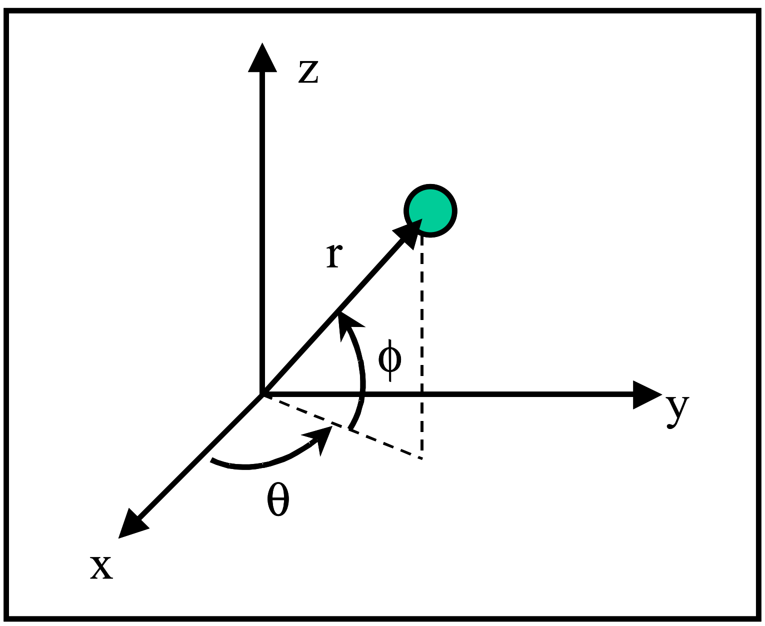
Degrees of Freedom (DOF)

- DOF = $n - m$
  - n = number of coordinates
  - m = number of constraints

Critical Point: The number of DOF is a characteristic of the system and does **NOT** depend on the particular set of coordinates used to describe the configuration.

Example 1

- Particle in space



$n = 3$

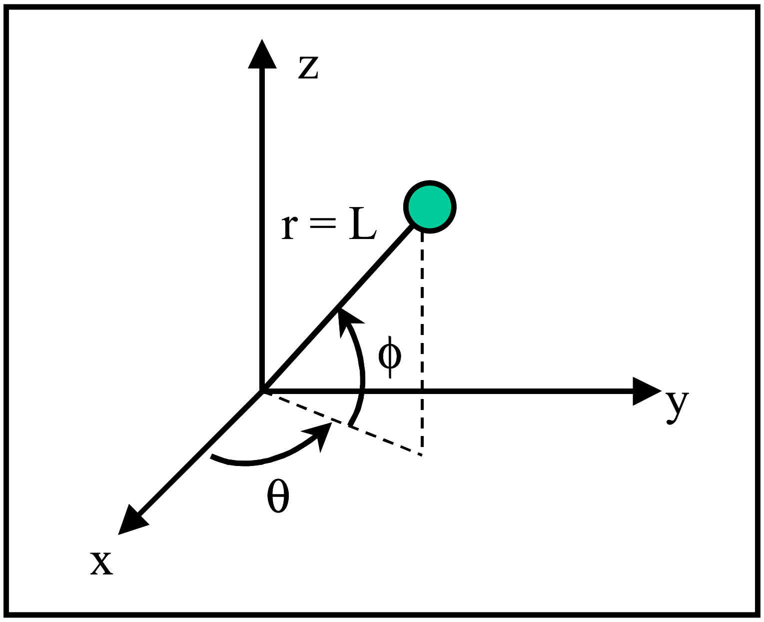
Coordinate sets: x, y, z *or* r, θ, ϕ

$m = 0$

DOF = $n - m = 3$

Example 2

- Conical Pendulum



Cartesian Coordinates	Spherical Coordinates
$n = 3\ (x, y, z)$	$n = 2\ (\theta, \phi)$
$m = 1\ (x^2 + y^2 + z^2 = R^2)$	$m = 0$
DOF = 2	DOF = 2

Example 3

- Two particles at a fixed distance (dumbbell)

Coordinates: ___________________

n = _____

m = _____

EOC's = __________________

DOF = ______

Generalized Coordinates

- No specific set of coordinates is required to analyze the system.
- Number of coordinates depends on the system, and not the set selected.
- Any set of parameters that are used to represent a system are called generalized coordinates.

Coordinate Transformation

- Often find that the "best" set of generalized coordinates used to solve a problem may not provide the information needed for further analysis.
- Use a **coordinate transformation** to convert between sets of generalized coordinates.

Example: Work in polar coordinates, then transform to rectangular coordinates, e.g.

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

General Form of the Transformation

Consider a system of N particles ➔ (Number of DOF = ___)

Let:

q_i be a set of generalized coordinates.

x_i be a set of Cartesian coordinates relative to an inertial frame

Transformation equations are:

$$
\begin{aligned}
x_1 &= f_1(q_1, q_2, q_3, \dots q_n, t) \\
x_2 &= f_2(q_1, q_2, q_3, \dots q_n, t) \\
&\vdots \\
x_n &= f_n(q_1, q_2, q_3, \dots q_n, t)
\end{aligned}
$$

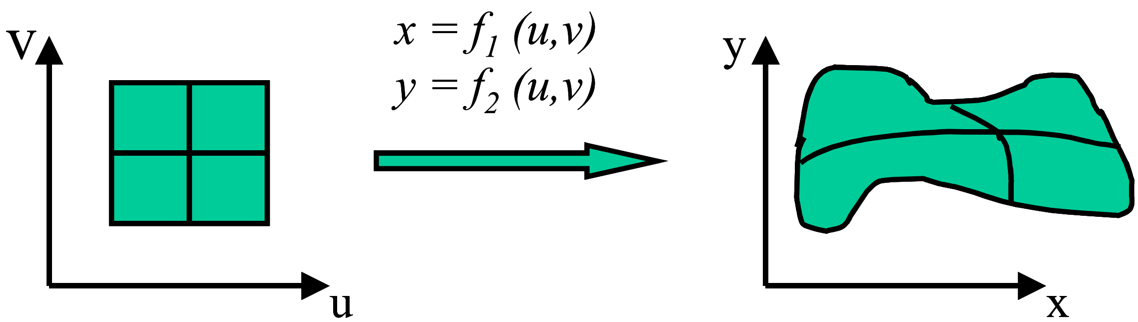
Each set of coordinates can have equations of constraint (EOC)

- Let l = number of EOC for the set of x_i
- Then DOF = $n - m = 3N - l$

Recall: **Number of generalized coordinates required depends on the system, not the set selected**.

Requirements for a coordinate transform

- Finite, single valued, continuous and differentiable
- Non-zero Jacobian $\mathrm{J} = \dfrac{\partial\left(x_1, x_2, x_3, \ldots x_n\right)}{\partial\left(q_1, q_2, q_3, \ldots q_n\right)}$
- No singular points



$$J = \begin{bmatrix} \dfrac{\partial x_1}{\partial q_1} & & \dfrac{\partial x_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x_n}{\partial q_1} & & \dfrac{\partial x_n}{\partial q_n} \end{bmatrix}$$

Example: Cartesian to Polar transformation

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \quad \Rightarrow \quad J = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

$$|J| = \cos\theta\left[r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right]$$
$$+r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right]$$

$$|J| = r^2\sin\theta \neq 0 \text{ for } r \neq 0 \text{ and } \theta \neq 0 \pm n\pi$$

Constraints

Existence of constraints complicates the solution of the problem.

- Can just eliminate the constraints
- Deal with them directly (Lagrange multipliers, more later).

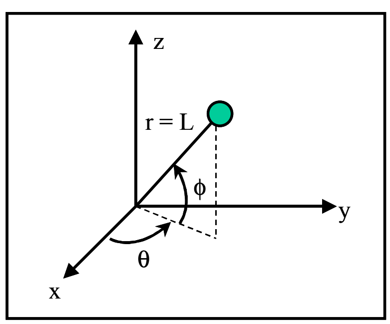
Holonomic Constraints can be expressed **algebraically**.

$$\phi_j(q_1, q_2, q_3, \ldots q_n, t) = 0, \; j = 1, 2, \ldots m$$

Properties of holonomic constraints

- Can always find a set of independent generalized coordinates
- Eliminate m coordinates to find $n - m$ independent generalized coordinates.

Example: Conical Pendulum



Cartesian Coordinates	Spherical Coordinates
$n = 3 \; (x, y, z)$	$n = (r, \theta, \phi)$
$m = 1 \; (x^2 + y^2 + z^2 = L^2)$	$m = 1, r = L$
DOF = 2	DOF = 2

Nonholonomic constraints cannot be written in a closed-form (algebraic equation), but instead must be expressed in terms of the differentials of the coordinates (and possibly time)

$$\sum_{i=1}^{n} a_{ji} dq_i + a_{jt} dt = 0, \; j = 1, 2, \ldots m$$

$$a_{ji} = \psi\left(q_1, q_2, q_3, \ldots q_n, t\right)$$

- Constraints of this type are non-integrable and restrict the velocities of the system.

$$\Rightarrow \; \sum_{i=1}^{n} a_{ji} \dot{q}_i + a_{jt} = 0, \; j = 1, 2, \ldots m$$

How determine if a differential equation is integrable and therefore holonomic?

- Integrable equations must be exact, i.e. they must satisfy the conditions: (i, k = 1,…,n)

$$\frac{\partial a_{ji}}{\partial q_k} = \frac{\partial a_{jk}}{\partial q_i}$$

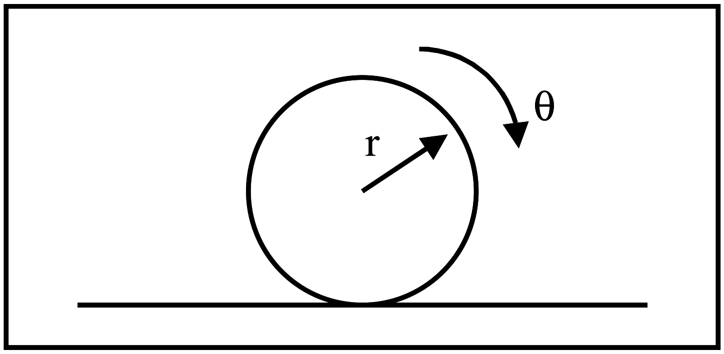
$$\frac{\partial a_{ji}}{\partial t} = \frac{\partial a_{jt}}{\partial q_i}$$

Key point: Nonholonomic constraints **do not** affect the number of DOF in a system.

Special cases of holonomic and nonholonomic constraints

- **Scleronomic** – No explicit dependence on t (time)
- **Rheonomic** – Explicit dependence on t

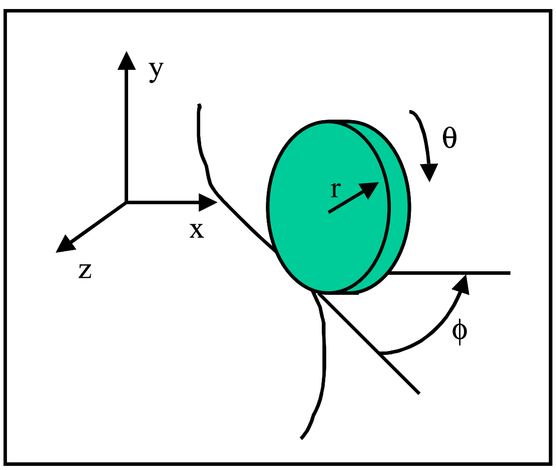
Example: Wheel rolling without slipping in a straight line



$$v = \dot{x} = r\dot{\theta}$$

$$dx - r d\theta = 0$$

Example: Wheel rolling without slipping on a curved path.
Define ϕ as angle between the tangent to the path and the x-axis.



$$\dot{x} = v\sin\phi = r\dot{\theta}\sin\phi$$

$$\dot{y} = v\cos\phi = r\dot{\theta}\cos\phi$$

$$dx - r\sin\phi\, d\theta = 0$$

$$dy - r\cos\phi\, d\theta = 0$$

Have 2 differential equations of constraint, neither of which can be integrated without solving the entire problem.
➔ Constraints are nonholonomic

Reason? Can relate change in θ to change in *x,y* for given ϕ, but the absolute value of θ depends on the path taken to get to that point (which is the "solution").

Summary to Date

Why use Lagrange Formulation?

1. Scalar, not vector
2. Eliminate solving for constraint forces
3. Avoid finding accelerations

DOF – Degrees of Freedom

- **DOF =** $\boldsymbol{n - m}$
- **n** is the number of coordinates
  – 3 for a particle
  – 6 for a rigid body
- **m** is the number of holonomic constraints

Generalized Coordinates q_i

- Term for any coordinate
- "Acquired skill" in applying Lagrange method is choosing a good set of generalized coordinates.

Coordinate Transform

- Mapping between sets of coordinates
- Non-zero Jacobian