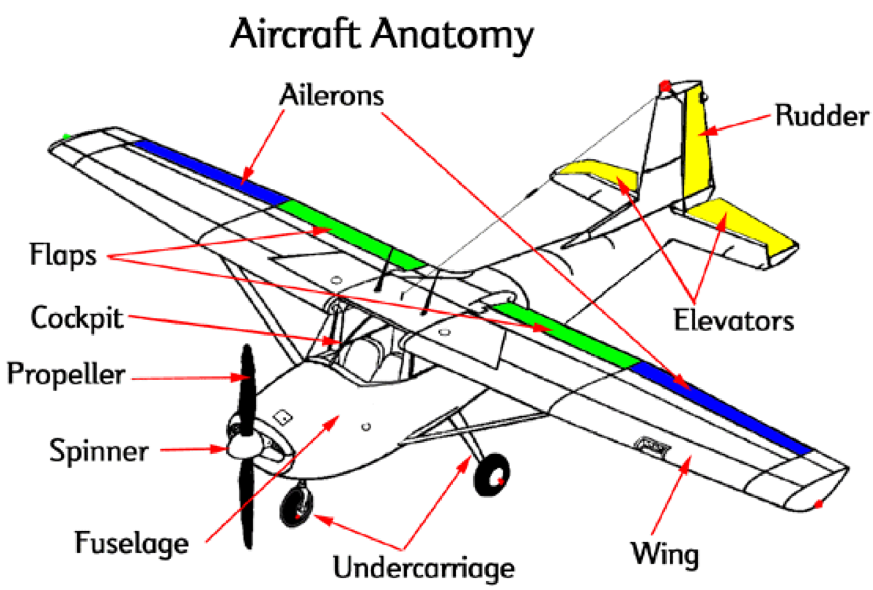
Lecture AC-1

Aircraft Dynamics



Aircraft Dynamics

- First note that it is possible to develop a very good approximation of a key motion of an aircraft (called the Phugoid mode) using a very simple balance between the kinetic and potential energies.
  - Consider an aircraft in steady, level flight with speed U_0 and height h_0. The motion is perturbed slightly so that

$$U_0 \rightarrow U = U_0 + u \quad (1)$$
$$h_0 \rightarrow h = h_0 + \Delta h \quad (2)$$

  - Assume that $E = \frac{1}{2}mU^2 + mgh$ is constant before and after the perturbation. It then follows that

$$u \approx -\frac{g\Delta h}{U_0}$$

  - From Newton's laws we know that, in the vertical direction

$$m\ddot{h} = L - W$$

    where weight $W = mg$ and lift $L = \frac{1}{2}\rho S C_L U^2$ (S is the wing area). We can then derive the equations of motion of the aircraft:

$$\begin{aligned} m\ddot{h} = L - W &= \frac{1}{2}\rho S C_L (U^2 - U_0^2) && (3) \\ &= \frac{1}{2}\rho S C_L ((U_0 + u)^2 - U_0^2) \approx \frac{1}{2}\rho S C_L (2uU_0) && (4) \\ &\approx -\rho S C_L \left(\frac{g\Delta h}{U_0} U_0\right) = -(\rho S C_L g)\Delta h && (5) \end{aligned}$$

    Since $\ddot{h} = \Delta\ddot{h}$ and for the original equilibrium flight condition $L = W = \frac{1}{2}(\rho S C_L)U_0^2 = mg$, we get that

$$\frac{\rho S C_L g}{m} = 2\left(\frac{g}{U_0}\right)^2$$

    Combine these result to obtain:

$$\Delta\ddot{h} + \Omega^2 \Delta h = 0 \quad , \quad \Omega \approx \frac{g}{U_0}\sqrt{2}$$

  - These equations describe an oscillation (called the phugoid oscillation) of the altitude of the aircraft about it nominal value.
    - ◇ Only approximate natural frequency, but value very close.

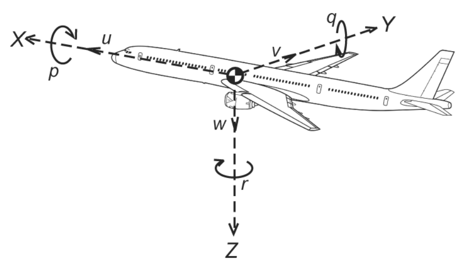
- The basic dynamics are the same as we had before:

$$\vec{F} = m\dot{\vec{v}}_c^{\,I} \quad \text{and} \quad \vec{T} = \dot{\vec{H}}^{I}$$

$$\Rightarrow \frac{1}{m}\vec{F} = \dot{\vec{v}}_c^{\,B} + {}^{BI}\vec{\omega} \times \vec{v}_c \qquad \textbf{Transport Thm.}$$

$$\Rightarrow \vec{T} = \dot{\vec{H}}^{B} + {}^{BI}\vec{\omega} \times \vec{H} \qquad \text{Note the notation change}$$

- Basic assumptions are:
  1. Earth is an inertial reference frame
  2. A/C is a rigid body
  3. Body frame **B** fixed to the aircraft $(\vec{i}, \vec{j}, \vec{k})$



- Instantaneous mapping of $\vec{v}_c$ and ${}^{BI}\vec{\omega}$ into the body frame is given by

$${}^{BI}\vec{\omega} = P\vec{i} + Q\vec{j} + R\vec{k} \qquad \vec{v}_c = U\vec{i} + V\vec{j} + W\vec{k}$$

$$\Rightarrow {}^{BI}\omega_B = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \qquad \Rightarrow (v_c)_B = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

- By symmetry, we can show that $I_{xy} = I_{yz} = 0$, but value of I_{xz} depends on specific frame selected. Instantaneous mapping of the angular momentum

$$\vec{H} = H_x\vec{i} + H_y\vec{j} + H_z\vec{k}$$

into the Body Frame given by

$$H_B = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{xz} & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

- The overall equations of motion are then:

$$\frac{1}{m}\vec{F} = \dot{\vec{v}}_c^{\,B} + {}^{BI}\vec{\omega} \times \vec{v}_c$$

$$\Rightarrow \frac{1}{m}\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix}\begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

$$= \begin{bmatrix} \dot{U} + QW - RV \\ \dot{V} + RU - PW \\ \dot{W} + PV - QU \end{bmatrix}$$

$$\vec{T} = \dot{\vec{H}}^{B} + {}^{BI}\vec{\omega} \times \vec{H}$$

$$\Rightarrow \begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{P} + I_{xz}\dot{R} \\ I_{yy}\dot{Q} \\ I_{zz}\dot{R} + I_{xz}\dot{P} \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix}\begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{xz} & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx}\dot{P} + I_{xz}\dot{R} & +QR(I_{zz} - I_{yy}) + PQI_{xz} \\ I_{yy}\dot{Q} & +PR(I_{xx} - I_{zz}) + (R^2 - P^2)I_{xz} \\ I_{zz}\dot{R} + I_{xz}\dot{P} & +PQ(I_{yy} - I_{xx}) - QRI_{xz} \end{bmatrix}$$

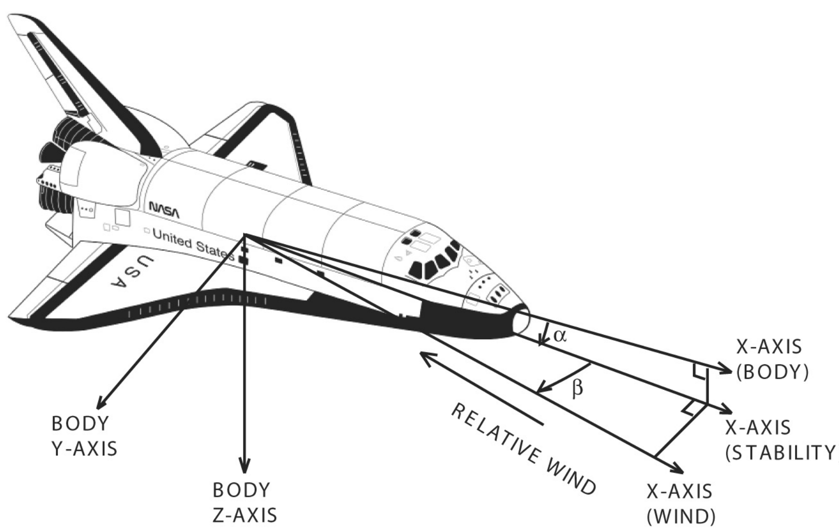
- Clearly these equations are very nonlinear and complicated, and we have not even said where $\vec{F}$ and $\vec{T}$ come from. $\Longrightarrow$ Need to linearize!!
  - Assume that the aircraft is flying in an *equilibrium condition* and we will linearize the equations about this nominal flight condition.

- But first we need to be a little more specific about which *Body Frame* we are going use. Several standards:

  1. **Body Axes** - X aligned with fuselage nose. Z perpendicular to X in plane of symmetry (down). Y perpendicular to XZ plane, to the right.

  2. **Wind Axes** - X aligned with $\vec{v}_c$. Z perpendicular to X (pointed down). Y perpendicular to XZ plane, off to the right.

  3. **Stability Axes** - X aligned with projection of $\vec{v}_c$ into the fuselage plane of symmetry. Z perpendicular to X (pointed down). Y same.



- Advantages to each, but typically use the **stability axes**.

  – In different *flight equilibrium conditions*, the axes will be oriented differently with respect to the A/C principal axes ⇒ need to transform (rotate) the principal Inertia components between the frames.

  – When vehicle undergoes motion with respect to the equilibrium, the **Stability Axes remain fixed to the airplane as if *painted on.***

- Can linearize about various steady state conditions of flight.
  - For steady state flight conditions must have

    $$\vec{F} = \vec{F}_{\text{aero}} + \vec{F}_{\text{gravity}} + \vec{F}_{\text{thrust}} = 0 \text{ and } \vec{T} = 0$$

    ◇ So for equilibrium condition, forces balance on the aircraft
    $L = W$ and $T = D$

  - Also assume that $\dot{P} = \dot{Q} = \dot{R} = \dot{U} = \dot{V} = \dot{W} = 0$

  - Impose additional constraints that depend on the **flight condition:**

    ◇ Steady wings-level flight $\rightarrow \Phi = \dot{\Phi} = \dot{\Theta} = \dot{\Psi} = 0$

- **Key Point:** While nominal forces and moments balance to zero, motion about the equilibrium condition results in perturbations to the forces/moments.
  - Recall from basic flight dynamics that lift $L_0^f = C_l \alpha_0$, where:

    ◇ C_l = *lift coefficient*, which is a function of the equilibrium condition

    ◇ α_0 = nominal *angle of attack* (angle that the wing meets the air flow).
  - But, as the vehicle moves about the equilibrium condition, would expect that the angle of attack will change

    $$\alpha = \alpha_0 + \Delta\alpha$$

  - Thus the lift forces will also be perturbed

    $$L^f = C_l(\alpha_0 + \Delta\alpha) = L_0^f + \Delta L^f$$

- Can extend this idea to all dynamic variables and how they influence all aerodynamic forces and moments

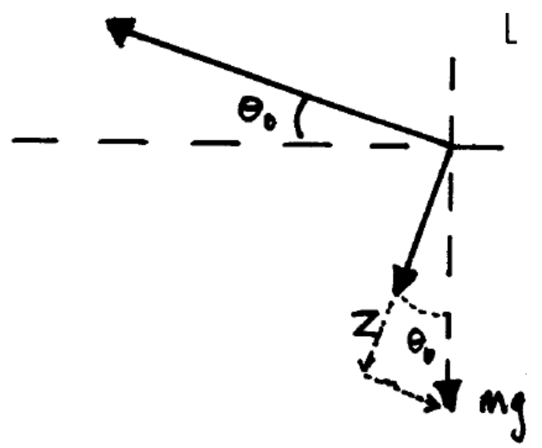
Gravity Forces

- Gravity acts through the CoM in vertical direction (inertial frame +Z)
  - Assume that we have a non-zero pitch angle Θ_0
  - Need to map this force into the body frame
  - Use the Euler angle transformation (2–15)

$$F_B^g = T_1(\Phi)T_2(\Theta)T_3(\Psi)\begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = mg \begin{bmatrix} -\sin\Theta \\ \sin\Phi\cos\Theta \\ \cos\Phi\cos\Theta \end{bmatrix}$$

- For symmetric steady state flight equilibrium, we will typically assume that $\Theta \equiv \Theta_0$, $\Phi \equiv \Phi_0 = 0$, so

$$F_B^g = mg \begin{bmatrix} -\sin\Theta_0 \\ 0 \\ \cos\Theta_0 \end{bmatrix}$$



- Use Euler angles to specify vehicle rotations with respect to the Earth frame

$$\begin{aligned} \dot\Theta &= Q\cos\Phi - R\sin\Phi \\ \dot\Phi &= P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta \\ \dot\Psi &= (Q\sin\Phi + R\cos\Phi)\sec\Theta \end{aligned}$$

  - Note that if $\Phi \approx 0$, then $\dot\Theta \approx Q$

- **Recall:** $\Phi \approx$ Roll, $\Theta \approx$ Pitch, and $\Psi \approx$ Heading.

Recall:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = T_3(\psi) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = T_2(\theta) \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = T_1(\phi) \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}$$

Linearization

- Define the **trim** angular rates and velocities

$$ {}^{BI}\omega_B^o = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \qquad (v_c)_B^o = \begin{bmatrix} U_o \\ 0 \\ 0 \end{bmatrix} $$

  which are associated with the flight condition. In fact, these define the type of equilibrium motion that we linearize about. **Note:**

  – $W_0 = 0$ since we are using the stability axes, and

  – $V_0 = 0$ because we are assuming symmetric flight

- Proceed with the linearization of the dynamics for various flight conditions

	Nominal Velocity	**Perturbed** Velocity	$\Rightarrow$	**Perturbed** Acceleration
Velocities	U_0,	$U = U_0 + u$	$\Rightarrow$	$\dot{U} = \dot{u}$
	$W_0 = 0$,	$W = w$	$\Rightarrow$	$\dot{W} = \dot{w}$
	$V_0 = 0$,	$V = v$	$\Rightarrow$	$\dot{V} = \dot{v}$
Angular Rates	$P_0 = 0$,	$P = p$	$\Rightarrow$	$\dot{P} = \dot{p}$
	$Q_0 = 0$,	$Q = q$	$\Rightarrow$	$\dot{Q} = \dot{q}$
	$R_0 = 0$,	$R = r$	$\Rightarrow$	$\dot{R} = \dot{r}$
Angles	Θ_0,	$\Theta = \Theta_0 + \theta$	$\Rightarrow$	$\dot{\Theta} = \dot{\theta}$
	$\Phi_0 = 0$,	$\Phi = \phi$	$\Rightarrow$	$\dot{\Phi} = \dot{\phi}$
	$\Psi_0 = 0$,	$\Psi = \psi$	$\Rightarrow$	$\dot{\Psi} = \dot{\psi}$

- **Linearization for symmetric flight** $U = U_0 + u$, $V_0 = W_0 = 0$, $P_0 = Q_0 = R_0 = 0$. Note that the forces and moments are also perturbed.

$$\frac{1}{m}\left[F_x^0 + \Delta F_x\right] = \dot{U} + QW - RV \approx \dot{u} + qw - rv \approx \dot{u}$$

$$\frac{1}{m}\left[F_y^0 + \Delta F_y\right] = \dot{V} + RU - PW \approx \dot{v} + r(U_0 + u) - pw \approx \dot{v} + rU_0$$

$$\frac{1}{m}\left[F_z^0 + \Delta F_z\right] = \dot{W} + PV - QU \approx \dot{w} + pv - q(U_0 + u) \approx \dot{w} - qU_0$$

$$\Rightarrow \frac{1}{m}\begin{bmatrix} \Delta F_x \\ \Delta F_y \\ \Delta F_z \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{v} + rU_0 \\ \dot{w} - qU_0 \end{bmatrix} \quad \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$$

- Attitude motion:

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{P} + I_{xz}\dot{R} & +QR(I_{zz} - I_{yy}) + PQI_{xz} \\ I_{yy}\dot{Q} & +PR(I_{xx} - I_{zz}) + (R^2 - P^2)I_{xz} \\ I_{zz}\dot{R} + I_{xz}\dot{P} & +PQ(I_{yy} - I_{xx}) - QRI_{xz} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \Delta L \\ \Delta M \\ \Delta N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{p} + I_{xz}\dot{r} \\ I_{yy}\dot{q} \\ I_{zz}\dot{r} + I_{xz}\dot{p} \end{bmatrix} \quad \begin{matrix} \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \end{matrix}$$

Key aerodynamic parameters are also perturbed:

Total Velocity	V_T	$=$	$((U_0 + u)^2 + v^2 + w^2)^{1/2} \approx U_0 + u$
Perturbed **Sideslip angle**	β	$=$	$\sin^{-1}(v/V_T) \approx v/U_0$
Perturbed **Angle of Attack**	α_x	$=$	$\tan^{-1}(w/U) \approx w/U_0$

- To understand these equations in detail, and the resulting impact on the vehicle dynamics, we must investigate the terms $\Delta F_x \ldots \Delta N$.

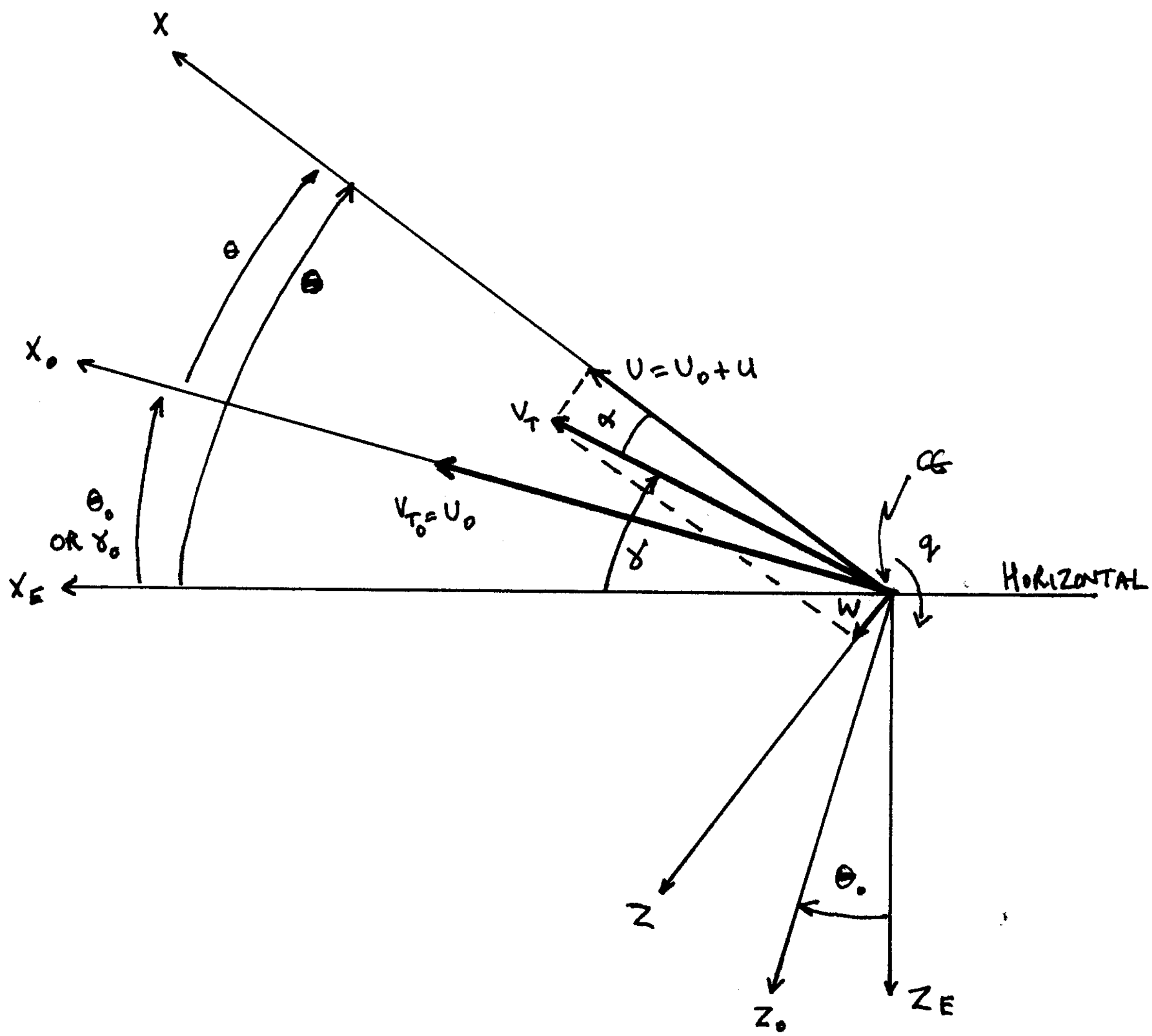


Figure 1: Perturbed Axes. The equilibrium condition was that the aircraft was angled up by Θ_0 with velocity $V_{T0} = U_0$. The vehicle's motion has been perturbed ($X_0 \to X$) so that now $\Theta = \Theta_0 + \theta$ and the velocity is $V_T \neq V_{T0}$. Note that V_T is no longer aligned with the X-axis, resulting in a non-zero u and w. The angle γ is called the **flight path angle,** and it provides a measure of the angle of the velocity vector to the inertial horizontal axis.

- We must also address the left-hand side $(\vec{F}, \vec{T})$
  - **Net** forces and moments must be zero in the equilibrium condition.
  - Aerodynamic and Gravity forces are a function of equilibrium condition **AND** the perturbations about this equilibrium.

- Predict the changes to the aerodynamic forces and moments using a first order expansion in the key flight parameters

$$\begin{aligned}\Delta F_x &= \frac{\partial F_x}{\partial U}\boldsymbol{\Delta U} + \frac{\partial F_x}{\partial W}\boldsymbol{\Delta W} + \frac{\partial F_x}{\partial \dot{W}}\boldsymbol{\Delta \dot{W}} + \frac{\partial F_x}{\partial \Theta}\Delta\Theta + \ldots + \frac{\partial F_x^g}{\partial \Theta}\Delta\Theta + \Delta F_x^c \\ &= \frac{\partial F_x}{\partial U}\mathbf{u} + \frac{\partial F_x}{\partial W}\mathbf{w} + \frac{\partial F_x}{\partial \dot{W}}\dot{\mathbf{w}} + \frac{\partial F_x}{\partial \Theta}\theta + \ldots + + \frac{\partial F_x^g}{\partial \Theta}\theta + \Delta F_x^c\end{aligned}$$

  - $\frac{\partial F_x}{\partial U}$ called a **stability derivative**. Is a function of the equilibrium condition. Usually tabulated.
  - Clearly an approximation since there tend to be lags in the aerodynamics forces that this approach ignores (assumes that forces only function of instantaneous values)
  - First proposed by Bryan (1911), and has proven to be a **very** effective way to analyze the aircraft flight mechanics – well supported by numerous flight test comparisons.

Stability Derivatives

- The forces and torques acting on the aircraft are very complex nonlinear functions of the flight equilibrium condition and the perturbations from equilibrium.
  - Linearized expansion can involve many terms $u, \dot{u}, \ddot{u}, \ldots, w, \dot{w}, \ddot{w}, \ldots$
  - Typically only retain a few terms to capture the dominant effects.

- Dominant behavior most easily discussed in terms of the:
  - Symmetric variables: U, W, Q and forces/torques: F_x, F_z, and M
  - Asymmetric variables: V, P, R and forces/torques: F_y, L, and N

- Observation – for truly symmetric flight Y, L, and N will be exactly **zero** for any value of U, W, Q

  $\Rightarrow$ Derivatives of asymmetric forces/torques with respect to the symmetric motion variables are **zero**.

- Further (convenient) assumptions:
  1. Derivatives of symmetric forces/torques with respect to the asymmetric motion variables are **zero**.
  2. We can neglect derivatives with respect to the derivatives of the motion variables, but keep $\partial F_z/\partial \dot{w}$ and $M_{\dot{w}} \equiv \partial M/\partial \dot{w}$ (aerodynamic lag involved in forming new pressure distribution on the wing in response to the perturbed angle of attack)
  3. $\partial F_x/\partial q$ is negligibly small.

- Note that we must also find the perturbation gravity and thrust forces and moments

$$\left.\frac{\partial F_x^g}{\partial \Theta}\right|_0 = -mg\cos\Theta_0 \qquad \left.\frac{\partial F_z^g}{\partial \Theta}\right|_0 = -mg\sin\Theta_0$$

- Typical set of stability derivatives.

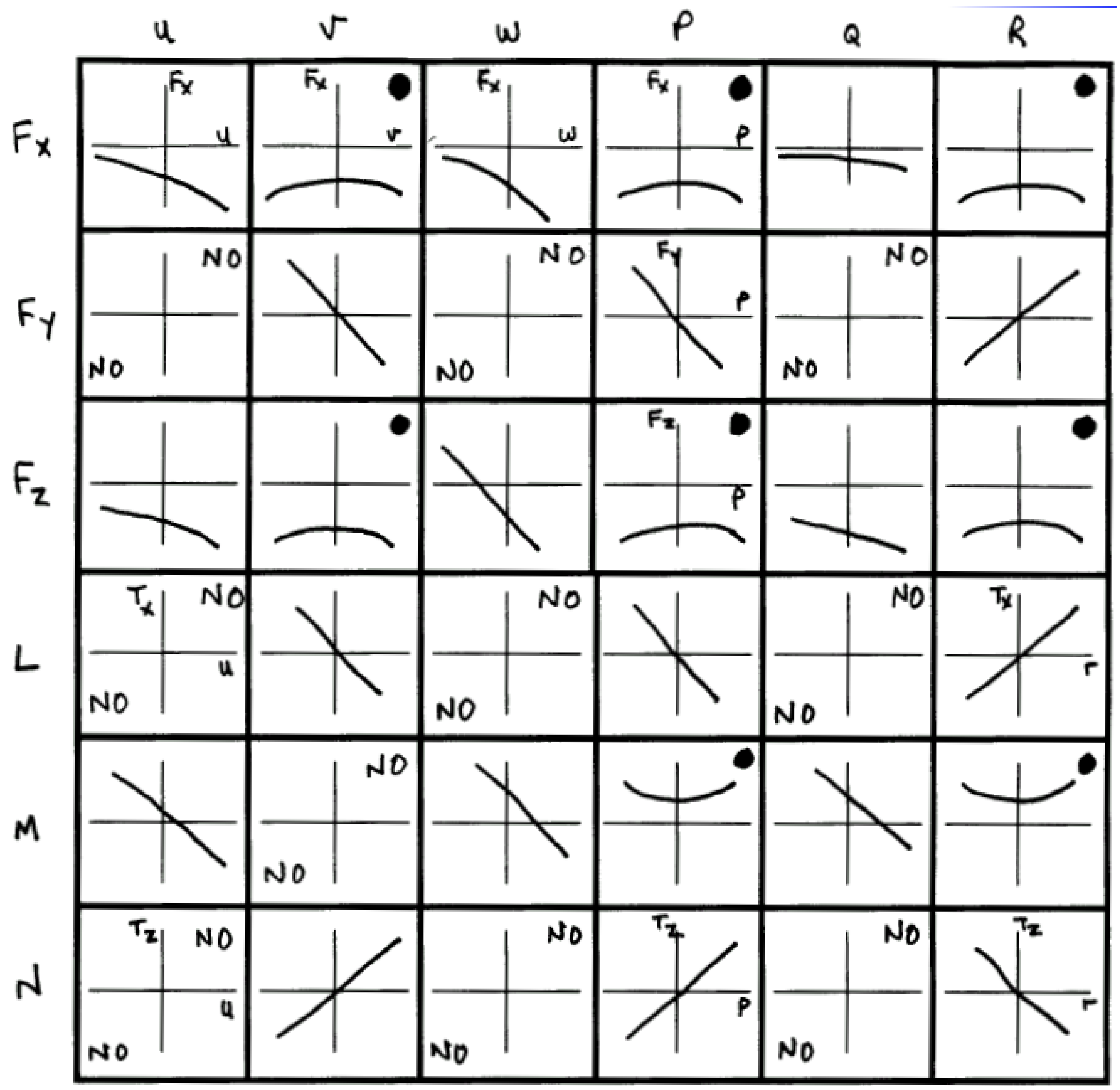


Figure 2: • corresponds to a zero slope - no dependence for small perturbations. **No** means no dependence for any size perturbation.

- **Aerodynamic summary:**

1A $\Delta F_x = \left(\frac{\partial F_x}{\partial U}\right)_0 u + \left(\frac{\partial F_x}{\partial W}\right)_0 w \Rightarrow \Delta F_x \sim u,\, w$

2A $\Delta F_y \sim v,\, p,\, r$

3A $\Delta F_z \sim u,\, w,\, \dot{w},\, q$

4A $\Delta L \sim \beta,\, p,\, r$

5A $\Delta M \sim u,\, w,\, \dot{w},\, q$

6A $\Delta N \sim \beta,\, p,\, r$

- Result is that, with these force, torque approximations, equations **1**, **3**, **5** decouple from **2 4**, **6**
  - **1**, **3**, **5** are the **longitudinal dynamics** in u, w, and q

$$\begin{bmatrix} \Delta F_x \\ \Delta F_z \\ \Delta M \end{bmatrix} = \begin{bmatrix} m\dot{u} \\ m(\dot{w} - qU_0) \\ I_{yy}\dot{q} \end{bmatrix}$$

$$\approx \begin{bmatrix} \left(\frac{\partial F_x}{\partial U}\right)_0 u + \left(\frac{\partial F_x}{\partial W}\right)_0 w + \left(\frac{\partial F_x^g}{\partial \Theta}\right)_0 \theta + \Delta F_x^c \\ \left(\frac{\partial F_z}{\partial U}\right)_0 u + \left(\frac{\partial F_z}{\partial W}\right)_0 w + \left(\frac{\partial F_z}{\partial \dot{W}}\right)_0 \dot{w} + \left(\frac{\partial F_z}{\partial Q}\right)_0 q + \left(\frac{\partial F_z^g}{\partial \Theta}\right)_0 \theta + \Delta F_z^c \\ \left(\frac{\partial M}{\partial U}\right)_0 u + \left(\frac{\partial M}{\partial W}\right)_0 w + \left(\frac{\partial M}{\partial \dot{W}}\right)_0 \dot{w} + \left(\frac{\partial M}{\partial Q}\right)_0 q + \Delta M^c \end{bmatrix}$$

  - **2**, **4**, **6** are the **lateral dynamics** in v, p, and r

Summary

- Picked a specific Body Frame (stability axes) from the list of alternatives

  ⇒ Choice simplifies some of the linearization, but the inertias now change depending on the equilibrium flight condition.

- Since the nonlinear behavior is too difficult to analyze, we needed to consider the linearized dynamic behavior around a specific flight condition

  ⇒ Enables us to linearize RHS of equations of motion.

- Forces and moments also complicated nonlinear functions, so we linearized the LHS as well

  ⇒ Enables us to write the perturbations of the forces and moments in terms of the motion variables.

  – Engineering insight allows us to argue that many of the stability derivatives that couple the longitudinal (symmetric) and lateral (asymmetric) motions are small and can be ignored.

- Approach requires that you have the stability derivatives.

  – These can be measured or calculated from the aircraft plan form and basic aerodynamic data.