Lecture #10

Friction in Lagrange's Formulation

Generalized Forces Revisited

- Derived Lagrange's Equation from D'Alembert's equation:

$$\sum_{i=1}^{p} m_i \left(\ddot{x}_i \delta x_i + \ddot{y}_i \delta y_i + \ddot{z}_i \delta z_i \right) = \sum_{i=1}^{p} \left(F_{x_i} \delta x_i + F_{y_i} \delta y_i + F_{z_i} \delta z_i \right)$$

- Define virtual displacements $\delta x_i = \sum_{j=1}^{N} \left(\frac{\partial x_i}{\partial q_j} \right) \delta q_j$
- Substitute in and noting the independence of the δq_j, for each DOF we get one Lagrange equation:

$$\sum_{i=1}^{p} m \left(\ddot{x}_i \frac{\partial x_i}{\partial q_r} + \ddot{y}_i \frac{\partial y_i}{\partial q_r} + \ddot{z}_i \frac{\partial z_i}{\partial q_r} \right) \delta q_r = \sum_{i=1}^{p} \left(F_{x_i} \frac{\partial x_i}{\partial q_r} + F_{y_i} \frac{\partial y_i}{\partial q_r} + F_{z_i} \frac{\partial z_i}{\partial q_r} \right) \delta q_r$$

- Applying lots of calculus on LHS and noting independence of the δq_i, for each DOF we get a Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = \sum_{i=1}^{p} \left(F_{x_i} \frac{\partial x_i}{\partial q_r} + F_{y_i} \frac{\partial y_i}{\partial q_r} + F_{z_i} \frac{\partial z_i}{\partial q_r} \right)$$

- Further, we "moved" the conservative forces (those derivable from a potential function to the LHS:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} = \sum_{i=1}^{p} \left(F_{x_i} \frac{\partial x_i}{\partial q_r} + F_{y_i} \frac{\partial y_i}{\partial q_r} + F_{z_i} \frac{\partial z_i}{\partial q_r} \right)$$

- Define Generalized Force:

$$Q_{q_r} = \sum_{i=1}^{p}\left(F_{x_i}\frac{\partial x_i}{\partial q_r} + F_{y_i}\frac{\partial y_i}{\partial q_r} + F_{z_i}\frac{\partial z_i}{\partial q_r} \right)$$

- Recall that the RHS was derived from the virtual work:

$$\boxed{Q_{q_r} = \frac{\delta W}{\delta q_r}}$$

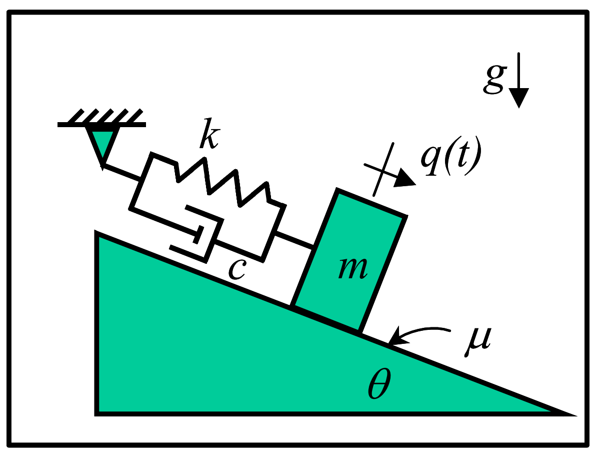
- Note, we can also find the effect of conservative forces using virtual work techniques as well.

<u>Example</u>

- Mass suspended from linear spring and velocity proportional damper slides on a plane with friction.

- Find the equation of motion of the mass.

- DOF = 3 – 2 = 1.



- Constraint equations: $y = z = 0$.

- Generalized coordinate: q

- Kinetic Energy: $T = \frac{1}{2}m\dot{q}^2$

- Potential Energy: $V = \frac{1}{2}kq^2 - mgq\sin\theta$

- Lagrangian: $L = T - V = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 + mgq\sin\theta$

- Derivatives:

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = m\ddot{q}, \quad \frac{\partial L}{\partial q} = -kq + mg\sin\theta$$

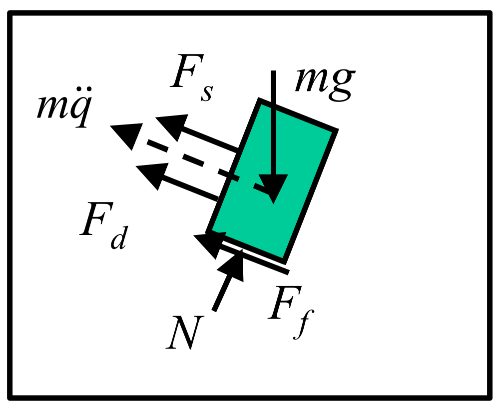
- Lagrange's Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = m\ddot{q} + kq - mg\sin\theta = Q_{q_r}$$

- To handle friction force in the generalized force term, need to know the normal force → Lagrange approach does not indicate the value of this force.

    - Look at the free body diagram.

    - Since body in motion at the time of the virtual displacement, use the d'Alembert principle and include the inertia forces as well as the real external forces



    - Sum forces perpendicular to the motion: $N = mg\cos\theta$

- Recall $\delta W = \mathbf{F} \cdot \delta \mathbf{s}$. Two nonconservative components, look at each component in turn:
  - Damper: $\delta W = -c\dot{q}\delta q$
  - Friction Force:

$$\begin{aligned} \delta W &= -\operatorname{sgn}(q)\mu N \delta q \\ &= -\operatorname{sgn}(q)\mu mg\cos\theta\delta q \end{aligned}$$

- Total Virtual Work:

$$\delta W = \left(-c\dot{q} - \operatorname{sgn}(q)\mu mg\cos\theta\right)\delta q$$

- The generalized force is thus:

$$Q_{q_r} = \frac{\delta W}{\delta q_r} = \left(-c\dot{q} - \operatorname{sgn}(q)\mu mg\cos\theta\right)$$

- And the EOM is:

$$m\ddot{q} + kq - mg\sin\theta = -c\dot{q} - \operatorname{sgn}(q)\mu mg\cos\theta$$

$$\Rightarrow m\ddot{q} + c\dot{q} + kq = mg\left(\sin\theta - \operatorname{sgn}(q)\mu\cos\theta\right)$$

- **Note:** Could have found the generalized forces using the coordinate system mapping:

$$Q_{q_r} = \sum_{i=1}^{p} \left(F_{x_i} \frac{\partial x_i}{\partial q_r} + F_{y_i} \frac{\partial y_i}{\partial q_r} + F_{z_i} \frac{\partial z_i}{\partial q_r} \right)$$

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  - For example, the gravity force:

$$F_{y_i} = -mg, \quad y_i = -q \sin\theta, \quad \frac{\partial y_i}{\partial q} = -\sin\theta$$

$$\Rightarrow Q_{q_r} = mg \sin\theta$$

Rayleigh's Dissipation Function

- For systems with conservative and non-conservative forces, we developed the general form of Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)-\frac{\partial L}{\partial q_r}=Q^N{}_{qr}$$

with L=T-V and

$$Q^N{}_{q_r}=F_x\frac{\partial x}{\partial q_r}+F_y\frac{\partial y}{\partial q_r}+F_z\frac{\partial z}{\partial q_r}$$

- For non-conservative forces that are a function of $\dot{q}$, there is an alternative approach. Consider generalized forces

$$Q^N{}_i=-\sum_{j=1}^{n}c_{ij}(q,t)\dot{q}_j$$

where the c_{ij} are the damping coefficients, which are dissipative in nature ➔ result in a loss of energy

- Now define the Rayleigh dissipation function

$$F=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}\dot{q}_i\dot{q}_j$$

- Then we can show that

$$\frac{\partial F}{\partial \dot{q}_r} = \sum_{j=1}^{n} c_{q_r j}\, \dot{q}_j = -Q^N{}_{q_r}$$

- So that we can rewrite Lagrange's equations in the slightly cleaner form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} + \frac{\partial F}{\partial \dot{q}_r} = 0$$

- In the example of the block moving on the wedge,

$$F = \frac{1}{2} c \dot{q}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial F}{\partial \dot{q}} = m\ddot{q} + kq - mg\sin\theta + c\dot{q} = Q'_{q_r}$$

where Q'_{q_r} now only accounts for the friction force.